Homework 5

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1 Sobolev norm

Let $H = C_2([0,1])$ be the set of twice continuously differentiable functions $f : [0,1] \to \mathbb{R}$, and $H_1 \subset H$ be the set of functions $f \in H$ that satisfy:

$$f(0) = f'(0) = 0.$$

Show that H_1 endowed with the norm:

$$\| f \|_{H_1}^2 = \int_0^1 f''(t)^2 dt$$

is a reproducing kernel Hilbert space (RKHS), and compute the reproducing kernel K_1.

2 B_n-splines

The convolution between two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f \star g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du,$$

when this integral exists.

Let now the function:

$$I(x) = \begin{cases} 1 & \text{si } -1 \leq x \leq 1, \\ 0 & \text{si } x < -1 \text{ ou } x > 1, \end{cases}$$

and $B_n = I^{\star n}$ for $n \in \mathbb{N}_*$ (that is, the function I convolved n times with itself: $B_1 = I, B_2 = I \star I, B_3 = I \star I \star I$, etc...).

Is the function $k(x,y) = B_n(x-y)$ a positive definite kernel over $\mathbb{R} \times \mathbb{R}$? If yes, describe the corresponding reproducing kernel Hilbert space.

3 More kernels...

3.1 Are the following functions positive definite kernels?

$$\forall x, y \in \mathbb{R}, \quad K_2(x, y) = \frac{1}{2 - e^{-\| x - y \|^2}}$$

$$\forall x, y \in \mathbb{R}, \quad K_3(x, y) = \max\left(0, 1 - |x - y|\right)$$

3.2. (BONUS) Can you describe the functions $\phi : \mathbb{R}^+ \mapsto \mathbb{R}$ such that:

$$K(x, y) = \phi\left(\max(x, y)\right)$$

is a positive definite kernel on $\mathbb{R}^+$?