Homework 1

Jean-Philippe Vert

Due January 21, 2010

1 Some kernels

Which of the following functions are positive definite:

\[
\begin{align*}
\forall -1 < x, y < 1 & \quad K_1(x, y) = \frac{1}{1 - xy} \\
\forall x, y \geq 0 & \quad K_2(x, y) = \min(x, y) \\
\forall x, y \geq 0 & \quad K_3(x, y) = \max(x, y) \\
\forall x, y > 0 & \quad K_4(x, y) = \frac{\min(x, y)}{\max(x, y)} \\
\forall x, y > 0 & \quad K_5(x, y) = \frac{\max(x, y)}{\min(x, y)} \\
\forall x, y \in \mathbb{R} & \quad K_6(x, y) = \cos(x + y) \\
\forall x, y \in \mathbb{R} & \quad K_7(x, y) = \cos(x - y) \\
\forall x, y \in \mathbb{R} & \quad K_8(x, y) = \sin(x + y) \\
\forall x, y \in \mathbb{R} & \quad K_9(x, y) = \sin(x - y)
\end{align*}
\]

2 Completeness of the RKHS

We want to finish the construction of the RKHS associated to a positive definite kernel \( K \) given in the course. Remember we have defined the set of functions:

\[
\mathcal{H}_0 = \left\{ \sum_{i=1}^{n} \alpha_i K_{x_i} : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, x_1, \ldots, x_n \in X \right\}
\]

1
and for any two functions $f, g \in \mathcal{H}_0$, given by:

\[ f = \sum_{i=1}^{m} a_i K_{x_i}, \quad g = \sum_{j=1}^{n} b_j K_{y_j}, \]

we have defined the operation:

\[ \langle f, g \rangle_{\mathcal{H}_0} := \sum_{i,j} a_i b_j K(x_i, y_j). \]

In the course we have shown that $\mathcal{H}_0$ endowed with this inner product is a pre-Hilbert space. Let us now show how to finish the construction of the RKHS from $\mathcal{H}_0$.

1. Show that any Cauchy sequence $(f_n)$ in $\mathcal{H}_0$ converges pointwisely to a function $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = \lim_{n \to +\infty} f_n(x)$.

2. Show that any Cauchy sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{H}_0$ which converges pointwise to 0 satisfies:

\[ \lim_{n \to +\infty} \|f_n\|_{\mathcal{H}_0} = 0. \]

3. Let $\mathcal{H} \subset \mathbb{R}^\mathcal{X}$ be the set of functions $f : \mathcal{X} \to \mathbb{R}$ which are pointwise limits of Cauchy sequences in $\mathcal{H}_0$, i.e., if $(f_n)$ is a Cauchy sequence in $\mathcal{H}_0$, then $f(x) = \lim_{n \to +\infty} f_n(x)$. Show that $\mathcal{H}_0 \subset \mathcal{H}$.

4. If $(f_n)$ and $(g_n)$ are two Cauchy sequences in $\mathcal{H}_0$, which converge pointwisely to two functions $f$ and $g \in \mathcal{H}$, show that the inner product $\langle f_n, g_n \rangle_{\mathcal{H}_0}$ converges to a number which only depends on $f$ and $g$. This allows us to define formally the operation:

\[ \langle f, g \rangle_{\mathcal{H}} = \lim_{n \to +\infty} \langle f_n, g_n \rangle_{\mathcal{H}_0}. \]

5. Show that $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner product on $\mathcal{H}$.

6. Show that $\mathcal{H}_0$ is dense in $\mathcal{H}$ (with respect to the metric defined by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$).

7. Show that $\mathcal{H}$ is complete.

8. Show that $\mathcal{H}$ is a RKHS whose reproducing kernel is $K$. 

2