1 Combining kernels

1. Let $K_1$ and $K_2$ be two positive definite kernels on a set $\mathcal{X}$. Show that the functions $K_1 + K_2$ and $K_1 \times K_2$ are also p.d. on $\mathcal{X}$.

2. Let $(K_i)_{i \geq 1}$ a sequence of p.d. kernel on a set $\mathcal{X}$ such that, for any $(x, y) \in \mathcal{X}^2$, the sequence $(K_i(x, y))_{i \geq 0}$ be convergent. Show that the pointwise limit:

$$K(x, y) = \lim_{i \to +\infty} K_i(x, y)$$

is also p.d.

2 Some kernels

Are the following functions positive definite?

$$\forall -1 < x, y < 1 \quad K_1(x, y) = \frac{1}{1 - xy}$$
$$\forall x, y \geq 0 \quad K_2(x, y) = \min(x, y)$$
$$\forall x, y \geq 0 \quad K_3(x, y) = \max(x, y)$$
$$\forall x, y > 0 \quad K_4(x, y) = \frac{\min(x, y)}{\max(x, y)}$$
$$\forall x, y \in \mathbb{R} \quad K_5(x, y) = \cos(x + y)$$
$$\forall x, y \in \mathbb{R} \quad K_6(x, y) = \cos(x - y)$$
3 Completeness of the RKHS

We want to finish the construction of the RKHS associated to a positive definite kernel $K$ given in the course. Remember we have defined the set of functions:

$$\mathcal{H}_0 = \left\{ \sum_{i=1}^{n} \alpha_i K_{x_i} : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, x_1, \ldots, x_n \in \mathcal{X} \right\}$$

and for any two functions $f, g \in \mathcal{H}_0$, given by:

$$f = \sum_{i=1}^{m} a_i K_{x_i}, \quad g = \sum_{j=1}^{n} b_j K_{y_j},$$

we have defined the operation:

$$\langle f, g \rangle_{\mathcal{H}_0} := \sum_{i,j} a_i b_j K(x_i, y_j).$$

In the course we have shown that $\mathcal{H}_0$ endowed with this inner product is a pre-Hilbert space. Let us now show how to finish the construction of the RKHS from $\mathcal{H}_0$.

1. Show that any Cauchy sequence $(f_n)$ in $\mathcal{H}_0$ converges pointwisely to a function $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = \lim_{n \to +\infty} f_n(x)$.

2. Show that any Cauchy sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{H}_0$ which converges pointwise to 0 satisfies:

$$\lim_{n \to +\infty} \| f_n \|_{\mathcal{H}_0} = 0.$$

3. Let $\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$ be the set of functions $f : \mathcal{X} \to \mathbb{R}$ which are pointwise limits of Cauchy sequences in $\mathcal{H}_0$, i.e., if $(f_n)$ is a Cauchy sequence in $\mathcal{H}_0$, then $f(x) = \lim_{n \to +\infty} f_n(x)$. Show that $\mathcal{H}_0 \subset \mathcal{H}$.

4. If $(f_n)$ and $(g_n)$ are two Cauchy sequences in $\mathcal{H}_0$, which converge pointwisely to two functions $f$ and $g \in \mathcal{H}$, show that the inner product $\langle f_n, g_n \rangle_{\mathcal{H}_0}$ converges to a number which only depends on $f$ and $g$. This allows us to define formally the operation:

$$\langle f, g \rangle_{\mathcal{H}} = \lim_{n \to +\infty} \langle f_n, g_n \rangle_{\mathcal{H}_0}.$$

5. Show that $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner product on $\mathcal{H}$.

6. Show that $\mathcal{H}_0$ is dense in $\mathcal{H}$ (with respect to the metric defined by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$).

7. Show that $\mathcal{H}$ is complete.

8. Show that $\mathcal{H}$ is a RKHS whose reproducing kernel is $K$. 
