# MVA "Kernel methods"
# Homework 1

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**Exercice 1.**

**1.** Let $K_1$ and $K_2$ be two positive definite (p.d.) kernels on a set $\mathcal{X}$. Show that the functions $K_1 + K_2$ and $K_1 \times K_2$ are also p.d. on $\mathcal{X}$.

**2.** Let $(K_i)_{i\geq 1}$ a sequence of p.d. kernel on a set $\mathcal{X}$ such that, for any $(x, y) \in \mathcal{X}^2$, the sequence $(K_i(x, y))_{i\geq 0}$ be convergent. Show that the pointwise limit:

$$K(x, y) = \lim_{i\to+\infty} K_i(x, y)$$

is also p.d. (assuming the limit exists for any $x, y$).

**3.** Are the following functions p.d.?

$$\begin{aligned}
\forall -1 < x, y < 1 \quad & K_1(x, y) = \frac{1}{1 - xy} \\
\forall x, y > 0 \quad & K_4(x, y) = \frac{\min(x, y)}{\max(x, y)} \\
\forall x, y \in \mathbb{R} \quad & K_5(x, y) = \cos(x + y) \\
\forall x, y \in \mathbb{R} \quad & K_6(x, y) = \cos(x - y)
\end{aligned}$$

**4.** Let $(\Omega, \mathcal{A}, P)$ be a probability space. Is the function

$$K : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$$

defined by:

$$\forall (A, B) \in \mathcal{A}^2, \quad K(A, B) = P(A \cap B) - P(A)P(B)$$

a p.d. kernel?

**Exercice 2.**
Prove that for any p.d. kernel $K$ on a space $\mathcal{X}$, a function $f : \mathcal{X} \to \mathbb{R}$ belongs to the RKHS $\mathcal{H}$ with kernel $K$ if and only if there exists $\lambda > 0$ such that $K(\mathbf{x}, \mathbf{x}') - \lambda f(\mathbf{x}) f(\mathbf{x}')$ is p.d.

**Exercice 3.**
Prove that if $K : \mathcal{X} \times \mathcal{X}$ is a positive definite function, then it is the r.k. of a unique RKHS. (Hint: consider the linear space spanned by the functions $K_x : t \mapsto K(x, t)$, and use the fact that a linear subspace $\mathcal{F}$ of a Hilbert space $\mathcal{H}$ is dense in $\mathcal{H}$ if and only $0$ is the only vector orthgonal to all vectors in $\mathcal{F}$)