Exercice 1.
The convolution between two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f \star g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du,$$

when this integral exists.

Let now the function:

$$I(x) = \begin{cases} 1 & \text{si } -1 \leq x \leq 1, \\ 0 & \text{si } x < -1 \text{ ou } x > 1, \end{cases}$$

and $B_n = I^*n$ for $n \in \mathbb{N}$ (that is, the function $I$ convolved $n$ times with itself:

$B_1 = I, B_2 = I \star I, B_3 = I \star I \star I$, etc...).

Is the function $k(x, y) = B_n(x - y)$ a positive definite kernel over $\mathbb{R} \times \mathbb{R}$? If yes, describe the corresponding reproducing kernel Hilbert space.

Exercice 2.
Are the following functions positive definite kernels?

$$\forall x, y \in \mathbb{R}, \quad K_1(x, y) = \frac{1}{2} e^{-\|x-y\|^2}$$

$$\forall x, y \in \mathbb{R}, \quad K_2(x, y) = \max (0, 1 - |x - y|)$$
\[ \forall x, y \in \mathbb{R}^+, \quad K_3(x, y) = \frac{1}{1 + x + y} \]

**Exercice 3.**
Describe the functions \( \phi : [0, 1] \mapsto \mathbb{R} \) such that:

\[ K(x, y) = \phi(\max(x + y - 1, 0)) \]

is a positive definite kernel on \([0, 1]\).

**Exercice 4.**
Describe the functions \( \phi : \mathbb{R}^+ \mapsto \mathbb{R} \) such that:

\[ K(x, y) = \phi(\max(x, y)) \]

is a positive definite kernel on \( \mathbb{R}^+ \).