Exercice 1.
1. Let $K_1$ and $K_2$ be two positive definite (p.d.) kernels on a set $\mathcal{X}$. Show that the functions $K_1 + K_2$ and $K_1 \times K_2$ are also p.d. on $\mathcal{X}$.
2. Let $(K_i)_{i \geq 1}$ a sequence of p.d. kernel on a set $\mathcal{X}$ such that, for any $(x, y) \in \mathcal{X}^2$, the sequence $(K_i(x, y))_{i \geq 0}$ be convergent. Show that the pointwise limit:
   \[ K(x, y) = \lim_{i \to +\infty} K_i(x, y) \]
is also p.d. (assuming the limit exists for any $x, y$).
3. Show that the following kernel is p.d.:
   \[ \forall -1 < x, y < 1 \quad K_1(x, y) = \frac{1}{1 - xy} \]

Exercice 2.
Let $\Omega$ be a finite set of cardinality $|\Omega| = n$. Show that the following kernel defined on the set of subsets of $\Omega$ is p.d.:
\[ \forall A, B \subset \Omega \quad K(A, B) = \frac{|A \cap B|}{|A \cup B|} \]

Exercice 3.
Show that the kernel $K(x, y) = \min(x, y)$ is p.d. on $[0, 1]^2$. Describe its RKHS.
Exercice 4.
Prove that if $K : \mathcal{X} \times \mathcal{X}$ is a positive definite function, then it is the r.k. of a unique RKHS. (Hint: consider the linear space spanned by the functions $K_x : t \mapsto K(x, t)$, and use the fact that a linear subspace $\mathcal{F}$ of a Hilbert space $\mathcal{H}$ is dense in $\mathcal{H}$ if and only 0 is the only vector orthogonal to all vectors in $\mathcal{F}$)