Exercice 1.

Given two sets of real numbers \(X = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(Y = (y_1, \ldots, y_n) \in \mathbb{R}^n\), the covariance between \(X\) and \(Y\) is defined as

\[
\text{cov}_n(X, Y) = E_n(XY) - E_n(X)E_n(Y),
\]

where \(E_n(U) = (\sum_{i=1}^{n} u_i)/n\). The covariance is useful to detect linear relationships between \(X\) and \(Y\). In order to extend this measure to potential nonlinear relationships between \(X\) and \(Y\), we consider the following criterion:

\[
C^K_n(X, Y) = \max_{f,g \in B_K} \text{cov}_n(f(X), g(Y)),
\]

where \(K\) is a positive definite kernel on \(\mathbb{R}\), \(B_K\) is the unit ball of the RKHS of \(K\), and \(f(U) = (f(u_1), \ldots, f(u_n))\) for a vector \(U = (u_1, \ldots, u_n)\).

1. Express simply \(C^K_n(X, Y)\) for the linear kernel \(K(a, b) = ab\).
2. For a general kernel \(K\), express \(C^K_n(X, Y)\) in terms of the Gram matrices of \(X\) and \(Y\).