Exercice 1. Combining kernels.
1. Let $K_1$ and $K_2$ be two positive definite (p.d.) kernels on a set $\mathcal{X}$. Show that the functions $K_1 + K_2$ and $K_1 \times K_2$ are also p.d. on $\mathcal{X}$.
2. Let $(K_i)_{i \geq 1}$ a sequence of p.d. kernel on a set $\mathcal{X}$ such that, for any $(x, y) \in \mathcal{X}^2$, the sequence $(K_i(x, y))_{i \geq 0}$ be convergent. Show that the pointwise limit:

$$K(x, y) = \lim_{i \to +\infty} K_i(x, y)$$

is also p.d. (assuming the limit exists for any $x, y$).
3. Show that the following kernel is p.d.:

$$\forall x, y \in \mathbb{R} \quad K(x, y) = 3^{xy}.$$

Exercice 2. Completeness of the RKHS.
We want to finish the construction of the RKHS associated to a positive definite kernel $K$ given in the course. Remember we have defined the set of functions:

$$\mathcal{H}_0 = \left\{ \sum_{i=1}^{n} \alpha_i K_{x_i} : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, x_1, \ldots, x_n \in \mathcal{X} \right\}$$

and for any two functions $f, g \in \mathcal{H}_0$, given by:

$$f = \sum_{i=1}^{m} a_i K_{x_i}, \quad g = \sum_{j=1}^{n} b_j K_{y_j},$$

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we have defined the operation:

\[ \langle f, g \rangle_{H_0} := \sum_{i,j} a_i b_j K(x_i, y_j). \]

In the course we have shown that \( H_0 \) endowed with this inner product is a pre-
Hilbert space. Let us now show how to finish the construction of the RKHS from \( H_0 \)

1. Show that any Cauchy sequence \( (f_n) \) in \( H_0 \) converges pointwisely to a
   function \( f : \mathcal{X} \to \mathbb{R} \) defined by \( f(x) = \lim_{n \to +\infty} f_n(x) \).

2. Show that any Cauchy sequence \( (f_n)_{n \in \mathbb{N}} \) in \( H_0 \) which converges pointwise
   to 0 satisfies:
   \[ \lim_{n \to +\infty} \| f_n \|_{H_0} = 0. \]

3. Let \( \mathcal{H} \subset \mathbb{R}^\mathcal{X} \) be the set of functions \( f : \mathcal{X} \to \mathbb{R} \) which are pointwise
   limits of Cauchy sequences in \( H_0 \), i.e., if \( (f_n) \) is a Cauchy sequence in \( H_0 \), then
   \( f(x) = \lim_{n \to +\infty} f_n(x) \). Show that \( H_0 \subset \mathcal{H} \).

4. If \( (f_n) \) and \( (g_n) \) are two Cauchy sequences in \( H_0 \), which converge point-
   wisely to two functions \( f \) and \( g \in \mathcal{H} \), show that the inner product \( \langle f_n, g_n \rangle_{H_0} \)
   converges to a number which only depends on \( f \) and \( g \). This allows us to define
   formally the operation:
   \[ \langle f, g \rangle_{\mathcal{H}} = \lim_{n \to +\infty} \langle f_n, g_n \rangle_{H_0}. \]

5. Show that \( \langle ., . \rangle_{\mathcal{H}} \) is an inner product on \( \mathcal{H} \).

6. Show that \( H_0 \) is dense in \( \mathcal{H} \) (with respect to the metric defined by the inner
   product \( \langle ., . \rangle_{\mathcal{H}} \))

7. Show that \( \mathcal{H} \) is complete.

8. Show that \( \mathcal{H} \) is a RKHS whose reproducing kernel is \( K \).

**Exercice 3. Uniqueness of the RKHS**

Prove that if \( K : \mathcal{X} \times \mathcal{X} \) is a positive definite function, then it is the r.k. of a
unique RKHS. To prove it, you can consider two possible RKHS \( \mathcal{H} \) and \( \mathcal{H}' \), and
show that (i) they contain the same elements and (ii) their inner products are the
same. (Hint: consider the linear space spanned by the functions \( K_x : t \mapsto K(x,t) \),
and use the fact that a linear subspace \( F \) of a Hilbert space \( \mathcal{H} \) is dense in \( \mathcal{H} \) if and
only 0 is the only vector orthogonal to all vectors in \( F \))