Exercice 1.

Let \((x_1, y_1), \ldots, (x_n, y_n)\) a training set of examples where \(x_i \in \mathcal{X}\), a space endowed with a positive definite kernel \(K\), and \(y_i \in \{-1, 1\}\), for \(i = 1, \ldots, n\). \(\mathcal{H}_K\) denotes the RKHS of the kernel \(K\). We want to learn a function \(f : \mathcal{X} \mapsto \mathbb{R}\) by solving the following optimization problem:

\[
\min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^{n} \ell_{y_i}(f(x_i)) \quad \text{such that} \quad \|f\|_{\mathcal{H}_K} \leq B, \tag{1}
\]

where \(\ell_y\) is a convex loss function (for \(y \in \{-1, 1\}\)) and \(B > 0\) is a parameter.

a. Show that there exists \(\lambda \geq 0\) such that the solution to problem (1) can be found by solving the following problem:

\[
\min_{\alpha \in \mathbb{R}^n} R(K\alpha) + \lambda \alpha^\top K \alpha, \tag{2}
\]

where \(K\) is the \(n \times n\) Gram matrix and \(R : \mathbb{R}^n \mapsto \mathbb{R}\) should be explicated.

b. Compute the Fenchel-Legendre transform\(^1\) \(R^*\) of \(R\) in terms of the Fenchel-Legendre transform \(\ell_y^*\) of \(\ell_y\).

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\(^1\)For any function \(f : \mathbb{R}^n \mapsto \mathbb{R}\), the Fenchel-Legendre transform (or convex conjugate) of \(f\) is the function \(f^* : \mathbb{R}^n \mapsto \mathbb{R}\) defined by

\[
f^*(u) = \sup_{x \in \mathbb{R}^n} \langle x, u \rangle - f(x).
\]
c. Adding the slack variable $u = K\alpha$, the problem (1) can be rewritten as a constrained optimization problem:

$$\min_{\alpha \in \mathbb{R}^n, u \in \mathbb{R}^n} R(u) + \lambda^\top K\alpha \quad \text{such that} \quad u = K\alpha.$$  \hspace{1cm} (3)

Express the dual problem of (3) in terms of $R^*$, and explain how a solution to (3) can be found from a solution to the dual problem.

c. Explicit the dual problem for the hinge loss:

$$\ell_y(u) = \max(0, 1 - yu).$$

**Exercice 2.**

Let $K_1$ and $K_2$ be two positive definite kernels on a set $\mathcal{X}$. What is the RKHS of the kernel $K_1 + K_2$?