Exercice 1. Kernel examples
Are the following kernels positive definite?

1. \( \forall x, y \in \mathbb{R} \quad K_1(x, y) = 10^{xy} , \quad K_2(x, y) = 10^{x+y} . \)

2. \( \forall x, y \in [0, 1) \quad K_3(x, y) = -\log(1 - xy) . \)

3. Let \( \mathcal{X} \) be a set and \( f, g : \mathcal{X} \to \mathbb{R}_+ \) two non-negative functions:

\[ \forall x, y \in \mathcal{X} \quad K_4(x, y) = \min(f(x)g(y), f(y)g(x)) \]

Exercice 2. Combining kernels

1. For \( x, y \in \mathbb{R} \), let

\[ K_1(x, y) = (xy + 1)^2 \quad \text{and} \quad K_2(x, y) = (xy - 1)^2 . \]

What is the RKHS of \( K_1 \)? Of \( K_2 \)? Of \( K_1 + K_2 \)?

2. Let \( K_1 \) and \( K_2 \) be two positive definite kernels on a set \( \mathcal{X} \), and \( \alpha, \beta \) two positive scalars. Show that \( \alpha K_1 + \beta K_2 \) is positive definite, and describe its RKHS.
Exercice 3. Uniqueness of the RKHS
Prove that if $K : \mathcal{X} \times \mathcal{X}$ is a positive definite function, then it is the r.k. of a unique RKHS. To prove it, you can consider two possible RKHS $\mathcal{H}$ and $\mathcal{H}'$, and show that (i) they contain the same elements and (ii) their inner products are the same. (Hint: consider the linear space spanned by the functions $K_x : t \mapsto K(x, t)$, and use the fact that a linear subspace $\mathcal{F}$ of a Hilbert space $\mathcal{H}$ is dense in $\mathcal{H}$ if and only 0 is the only vector orthogonal to all vectors in $\mathcal{F}$)