Exercice 1. Kernels

Show that the following kernels are positive definite:

1. On $\mathcal{X} = \mathbb{R}$:
   \[ \forall x, y \in \mathbb{R}, \quad K(x, y) = \cos(x - y). \]

2. On $\mathcal{X} = \{x \in \mathbb{R}^p : \|x\|_2 < 1\}$:
   \[ \forall x, y \in \mathcal{X}, \quad K(x, y) = 1/(1 - x^\top y). \]

3. Given a probability space $(\Omega, \mathcal{A}, P)$, on $\mathcal{X} = \mathbb{R}$:
   \[ \forall A, B \in \mathcal{A}, \quad K(A, B) = P(A \cap B) - P(A)P(B). \]

4. Let $\mathcal{X}$ be a set and $f, g : \mathcal{X} \to \mathbb{R}_+$ two non-negative functions:
   \[ \forall x, y \in \mathcal{X}, \quad K_4(x, y) = \min(f(x)g(y), f(y)g(x)) \]

5. Given a non-empty finite set $E$, on $\mathcal{X} = \mathcal{P}(E) = \{A : A \subset E\}$:
   \[ \forall A, B \subset E, \quad K(A, B) = \frac{|A \cap B|}{|A \cup B|}, \]

where $|F|$ denotes the cardinality of $F$, and with the convention $\frac{0}{0} = 0$.  

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Exercice 2. RKHS

1. Let $K_1$ and $K_2$ be two positive definite kernels on a set $\mathcal{X}$, and $\alpha, \beta$ two positive scalars. Show that $\alpha K_1 + \beta K_2$ is positive definite, and describe its RKHS.

2. Let $\mathcal{X}$ be a set and $\mathcal{F}$ be a Hilbert space. Let $\Psi : \mathcal{X} \to \mathcal{F}$, and $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be:

$$\forall x, x' \in \mathcal{X}, \quad K(x, x') = \langle \Psi(x), \Psi(x') \rangle_\mathcal{H}.$$ 

Show that $K$ is a positive definite kernel on $\mathcal{X}$, and describe its RKHS.

Exercice 3. Sobolev spaces

1. Let

$$\mathcal{H} = \left\{ f : [0, 1] \to \mathbb{R}, \text{ absolutely continuous}, f' \in L^2([0, 1]), f(0) = 0 \right\},$$

endowed with the bilinear form

$$\forall f, g \in \mathcal{H}, \quad \langle f, g \rangle_\mathcal{H} = \int_{0}^{1} f'(u)g'(u)du.$$ 

Show that $\mathcal{H}$ is an RKHS, and compute its reproducing kernel.

2. Same question when

$$\mathcal{H} = \left\{ f : [0, 1] \to \mathbb{R}, \text{ absolutely continuous}, f' \in L^2([0, 1]), f(0) = f(1) = 0 \right\},$$

3. Same question, when $\mathcal{H}$ is endowed with the bilinear form:

$$\forall f, g \in \mathcal{H}, \quad \langle f, g \rangle_\mathcal{H} = \int_{0}^{1} (f(u)g(u) + f'(u)g'(u))du.$$
Exercice 4. Duality

Let \((x_1, y_1), \ldots, (x_n, y_n)\) a training set of examples where \(x_i \in \mathcal{X}\), a space endowed with a positive definite kernel \(K\), and \(y_i \in \{-1, 1\}\), for \(i = 1, \ldots, n\). \(\mathcal{H}_K\) denotes the RKHS of the kernel \(K\). We want to learn a function \(f : \mathcal{X} \mapsto \mathbb{R}\) by solving the following optimization problem:

\[
\min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^{n} \ell_{y_i}(f(x_i)) \quad \text{such that} \quad \|f\|_{\mathcal{H}_K} \leq B, \tag{1}
\]

where \(\ell_y\) is a convex loss function (for \(y \in \{-1, 1\}\)) and \(B > 0\) is a parameter.

a. Show that there exists \(\lambda \geq 0\) such that the solution to problem (1) can be found by solving the following problem:

\[
\min_{\alpha \in \mathbb{R}^n} R(K\alpha) + \lambda \alpha^\top K \alpha, \tag{2}
\]

where \(K\) is the \(n \times n\) Gram matrix and \(R : \mathbb{R}^n \mapsto \mathbb{R}\) should be explicited.

b. Compute the Fenchel-Legendre transform\(^1\) of \(R\) in terms of the Fenchel-Legendre transform \(\ell_y^*\) of \(\ell_y\).

c. Adding the slack variable \(u = K\alpha\), the problem (1) can be rewritten as a constrained optimization problem:

\[
\min_{\alpha \in \mathbb{R}^n, u \in \mathbb{R}^n} R(u) + \lambda \alpha^\top K \alpha \quad \text{such that} \quad u = K\alpha. \tag{3}
\]

Express the dual problem of (3) in terms of \(R^*\), and explain how a solution to (3) can be found from a solution to the dual problem.

d. Explicit the dual problem for the logistic and squared hinge losses:

\[
\ell_y(u) = \log(1 + e^{-yu}).
\]

\[
\ell_y(u) = \max(0, 1 - yu)^2.
\]

\(^1\)For any function \(f : \mathbb{R}^N \mapsto \mathbb{R}\), the Fenchel-Legendre transform (or \emph{convex conjugate}) of \(f\) is the function \(f^* : \mathbb{R}^N \mapsto \mathbb{R}\) defined by

\[
f^*(u) = \sup_{x \in \mathbb{R}^N} x^\top u - f(x).
\]