Exercice 1. Kernels
Study whether the following kernels are positive definite:

1. $\mathcal{X} = \mathbb{N}$, $K(x, x') = 2^{x + x'}$
2. $\mathcal{X} = \mathbb{R}$, $K(x, x') = \cos(x + x')$
3. $\mathcal{X} = \mathbb{R}$, $K(x, x') = \cos(x - x')$

Exercice 2. Function and kernel boundedness
Consider a p.d. kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $K(x, z) \leq b^2$ for all $x, z$ in $\mathcal{X}$. Show that $\|f\|_{\infty} = \sup_{x \in \mathcal{X}} |f(x)| \leq b$ for any function $f$ in the unit ball of the corresponding RKHS.

Exercice 3. Non-expansiveness of the Gaussian kernel
Consider the Gaussian kernel $K : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ such that for all pair of points $x, x'$ in $\mathbb{R}^p$,

$$K(x, x') = e^{-\frac{\alpha}{2} \|x - x\|^2},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^p$. Call $\mathcal{H}$ the RKHS of $K$ and consider its RKHS mapping $\varphi : \mathbb{R}^p \to \mathcal{H}$ such that $K(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{H}}$ for all $x, x'$ in $\mathbb{R}^p$. Show that

$$\|\varphi(x) - \varphi(x')\|_{\mathcal{H}} \leq \sqrt{\alpha} \|x - x\|.$$
The mapping is called non-expansive whenever $\alpha \leq 1$. 