Homework 4

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Conditionally positive definite kernels

Let $\mathcal{X}$ be a set. A function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is called *conditionally positive definite* (c.p.d.) if and only if it is symmetric and satisfies:

$$\sum_{i,j=1}^{n} a_i a_j k(x_i, x_j) \geq 0$$

for any $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in \mathcal{X}^n$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}^n$ with $\sum_{i=1}^{n} a_i = 0$.

1. Show that a positive definite (p.d.) function is c.p.d.

2. Is a constant function p.d.? Is it c.p.d.?

3. If $\mathcal{X}$ is a Hilbert space, then is $k(x, y) = -||x - y||^2$ p.d.? Is it c.p.d.?

4. Let $\mathcal{X}$ be a nonempty set, and $x_0 \in \mathcal{X}$ a point. For any function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, let $\tilde{k} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be the function defined by:

$$\tilde{k}(x, y) = k(x, y) - k(x_0, x) - k(x_0, y) + k(x_0, x_0).$$

Show that k is c.p.d. if and only if $\tilde{k}$ is p.d.

5. Show that if k is c.p.d., then the function $\exp(tk(x, y))$ is p.d. for all $t \geq 0$

6. Conversely, show that if the function $\exp(tk(x, y))$ is p.d. for any $t \geq 0$, then k is c.p.d.

7. Let μ be a positive measure on $\mathbb{R}^+$, and $g : D(\mu) \to \mathbb{R}$ be the function:

$$g(x) = \int_0^\infty \left(e^{\lambda x} - 1\right) d\mu(\lambda),$$

where $D(\mu) \subset \mathbb{R}^+$ is the set of values x for which this integral is convergent.

7.a. If k is a c.p.d. kernel on a set $\mathcal{X}$, and if $k(\mathcal{X} \times \mathcal{X}) \subset D(\mu)$, show that $g \circ k$ is also c.p.d. on $\mathcal{X}$. (Suggestion: study first the function $g(x) = e^{\lambda x} - 1$)

7.b. Show that if k is c.p.d. and if $x_0 \in \mathcal{X}$ then the kernel:

$$\hat{k}(x, y) = g\left(k(x, y) + k(x_0, x_0)\right) - g\left(k(x, x_0) + k(y, x_0)\right)$$

is p.d. (if all the terms are well defined). (Suggestion: show that if k is p.d., then $e^k - 1$ is also p.d.).

8. The Gamma function is defined for any $x > 0$ by:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

8.a. Show that for any $0 < \alpha < 1$ and $z \geq 0$,

$$z^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-\lambda z}\right) \frac{d\lambda}{\lambda^{\alpha+1}}.$$

8.b. Show that if k is c.p.d. on $\mathcal{X}$ and $k(x, x) \leq 0$ for any $x \in \mathcal{X}$, then $-(-k)^\alpha$ is well defined and c.p.d. for $0 \leq \alpha \leq 1$.

9. Let k be a c.p.d. kernel on $\mathcal{X}$ such that $k(x, x) = 0$ for any $x \in \mathcal{X}$. Show that there exists a Hilbert space $\mathcal{H}$ and a mapping $\Phi : \mathcal{X} \to \mathcal{H}$ such that, for any $x, y \in \mathcal{X}$,

$$k(x, y) = -||\Phi(x) - \Phi(y)||^2.$$

10. For $x, y \in \mathbb{R}$, let

$$k(x, y) = e^{-|x-y|^p}.$$

10.a. Show that if k is p.d., then $-|x - y|^p$ is c.p.d.

10.b. Show that if k is p.d. then $p \leq 2$.

10.c. Conversely, show that if $p \leq 2$ then k is p.d.

11 Show that the opposite of the shortest-path distance on a tree is c.p.d over the set of vertices (a tree is an undirected graph without loops. The shortest-path distance between two vertices is the number of edges of the unique path that connects them). Is the shortest-path distance over graphs c.p.d. in general?