Uniqueness of the RKHS

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Recall the definition of an RKHS:

**Definition 1.** Let $X$ be a set and $\mathcal{H} \subset \mathbb{R}^X$ be a class of functions forming a Hilbert space with inner product $\langle ., . \rangle_{\mathcal{H}}$. The function $K : X^2 \mapsto \mathbb{R}$ is called a reproducing kernel (r.k.) of $\mathcal{H}$ if

1. $\mathcal{H}$ contains all functions of the form
   \[ \forall x \in X, \quad K_x : t \mapsto K(x, t) . \]  
   (1)

2. For every $x \in X$ and $f \in \mathcal{H}$ the reproducing property holds:
   \[ f(x) = \langle f, K_x \rangle_{\mathcal{H}} . \]  
   (2)

If a r.k. exists, then $\mathcal{H}$ is called a reproducing kernel Hilbert space (RKHS).

Remember that an RKHS has the following property

**Theorem 1.** A Hilbert space of functions $\mathcal{H} \subset \mathbb{R}^X$ is a RKHS if and only if for any $x \in X$, the mapping $f \mapsto f(x)$ (from $\mathcal{H}$ to $\mathbb{R}$) is continuous.

Suppose a sequence of function $(f_n)_{n \in \mathbb{N}}$ converges in a RKHS to a function $f \in \mathcal{H}$. Then the functions $(f_n - f)$ converges to 0 in the RKHS sense, from which we deduce that $f_n(x) - f(x)$ also converges to 0 for any $x \in X$, by continuity of the evaluations functionals. This proves that:

**Corollary 1.** Convergence in a RKHS implies pointwise convergence on any point, i.e., if $f_n$ converges to $f \in \mathcal{H}$, then $f_n(x)$ converges to $f(x)$ for any $x \in X$.

We now detail the proof of the following result, due to ?, which shows that there is a one-to-one correspondance between RKHS and r.k. It allows us to talk about “the” RHKS associated to a r.k., and conversely to “the” r.k. associated to a RKHS.

**Theorem 2.**

1. If a r.k. exists for a Hilbert space $\mathcal{H} \subset \mathbb{R}^X$, then it is unique.

2. Conversely, if two RKHS have the same r.k., then they are equal.

**Proof.** To prove 1., let $\mathcal{H}$ be a RKHS with two r.k. kernels $K$ and $K'$. For any two points $x, y \in X$, we need to show that $K(x, y) = K'(x, y)$. By the first property of RKHS, we know that the functions $K_x$ and $K'_x$ are in $\mathcal{H}$, and using the second property we obtain:

\[
\| K_x - K'_x \|_{\mathcal{H}}^2 = \langle K_x - K'_x, K_x - K'_x \rangle_{\mathcal{H}} \\
= \langle K_x - K'_x, K_x \rangle_{\mathcal{H}} - \langle K_x - K'_x, K'_x \rangle_{\mathcal{H}} \\
= K_x(x) - K'_x(x) - K_x(x) + K'_x(x) \\
= 0 .
\]
being a Hilbert space, only the zero function has a norm equal to 0. This shows that \( K_x = K'_x \) as functions, and in particular that \( K_x(y) = K'_x(y) \), i.e., \( K(x, y) = K'(x, y) \). To prove the converse, let us first consider a RKHS \( \mathcal{H}_1 \) with r.k. \( K \). By definition of the r.k., we know that all the functions \( K_x \) for \( x \in X \) are in \( \mathcal{H}_1 \), therefore their linear span

\[
\mathcal{H}_0 = \left\{ \sum_{i=1}^{n} \alpha_i K_{x_i} : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, x_1, \ldots, x_n \in X \right\}
\]

is a subspace of \( \mathcal{H}_1 \). Now we observe that if \( f \in \mathcal{H}_1 \) is orthogonal to \( \mathcal{H}_0 \), then in particular it is orthogonal to \( K_x \) for any \( x \) which implies \( f(x) = \langle f, K_x \rangle_{\mathcal{H}_1} = 0 \), i.e., \( f = 0 \). In other words, \( \mathcal{H}_0 \) is dense in \( \mathcal{H}_1 \). Moreover the \( \mathcal{H}_1 \) norm for functions in \( \mathcal{H}_0 \) only depends on the r.k. \( K \), because it is given for a function \( f = \sum_{i=1}^{n} \alpha_i K_{x_i} \) in \( \mathcal{H}_0 \) by

\[
\| f \|_{\mathcal{H}_1}^2 = \sum_{i,j=1}^{n} \alpha_i \alpha_j \langle K_{x_i}, K_{x_j} \rangle_{\mathcal{H}_1} = \sum_{i,j=1}^{n} \alpha_i \alpha_j K(x_i, x_j).
\]

Suppose now that \( \mathcal{H}_2 \) is also a RKHS that admits \( K \) as r.k. Then by the same argument, the space \( \mathcal{H}_0 \) is dense in \( \mathcal{H}_2 \), and the \( \mathcal{H}_2 \) norm in \( \mathcal{H}_0 \) is given by (3). In particular, for any \( f \in \mathcal{H}_0 \), \( \| f \|_{\mathcal{H}_1} = \| f \|_{\mathcal{H}_2} \). Let now \( f \in \mathcal{H}_1 \). By density of \( \mathcal{H}_0 \) in \( \mathcal{H}_1 \), there is a sequence \( (f_n) \) in \( \mathcal{H}_0 \) such that \( \| f_n - f \|_{\mathcal{H}_1} \to 0 \). The converging sequence \( (f_n) \) is in particular a Cauchy sequence for the \( \mathcal{H}_1 \) norm, and since this norm coincides with the \( \mathcal{H}_2 \) norm on \( \mathcal{H}_0 \), \( (f_n) \) is also a Cauchy sequence for the \( \mathcal{H}_2 \) norm and converges in \( \mathcal{H}_2 \) to a function \( g \in \mathcal{H}_2 \). By Corollary \[ applied to both \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), we see that, for any \( x \in X \), \( \lim_{n \to +\infty} f_n(x) = f(x) = g(x) \). In other words, \( f = g \) and therefore \( f \in \mathcal{H}_2 \). This shows that \( \mathcal{H}_1 \subset \mathcal{H}_2 \) and, by symmetry of the argument, in fact that \( \mathcal{H}_1 = \mathcal{H}_2 \). We now need to check that the norms in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) coincide, which results from:

\[
\| f \|_{\mathcal{H}_1} = \lim_{n \to +\infty} \| f_n \|_{\mathcal{H}_1} = \lim_{n \to +\infty} \| f_n \|_{\mathcal{H}_2} = \| f \|_{\mathcal{H}_2}.
\]