Uniqueness of the RKHS

Jean-Philippe Vert

Recall the definition of an RKHS:

**Definition 1.** Let $X$ be a set and $\mathcal{H} \subset \mathbb{R}^X$ be a class of functions forming a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The function $K : X^2 \mapsto \mathbb{R}$ is called a reproducing kernel (r.k.) of $\mathcal{H}$ if

1. $\mathcal{H}$ contains all functions of the form
   $$\forall x \in X, \quad K_{x} : t \mapsto K(x, t).$$

2. For every $x \in X$ and $f \in \mathcal{H}$ the reproducing property holds:
   $$f(x) = \langle f, K_x \rangle_{\mathcal{H}}.$$  

If a r.k. exists, then $\mathcal{H}$ is called a reproducing kernel Hilbert space (RKHS).

Remember that an RKHS has the following property

**Theorem 1.** A Hilbert space of functions $\mathcal{H} \subset \mathbb{R}^X$ is a RKHS if and only if for any $x \in X$, the mapping $f \mapsto f(x)$ (from $\mathcal{H}$ to $\mathbb{R}$) is continuous.

Suppose a sequence of function $(f_n)_{n \in \mathbb{N}}$ converges in a RKHS to a function $f \in \mathcal{H}$. Then the functions $(f_n - f)$ converges to 0 in the RKHS sense, from which we deduce that $f_n(x) - f(x)$ also converges to 0 for any $x \in X$, by continuity of the evaluations functionals. This proves that:

**Corollary 1.** Convergence in a RKHS implies pointwise convergence on any point, i.e., if $f_n$ converges to $f \in \mathcal{H}$, then $f_n(x)$ converges to $f(x)$ for any $x \in X$.

We now detail the proof of the following result, due to [1], which shows that there is a one-to-one correspondence between RKHS and r.k. It allows us to talk about "the" RHKS associated to a r.k., and conversely to "the" r.k. associated to a RKHS.

**Theorem 2.**

1. If a r.k. exists for a Hilbert space $\mathcal{H} \subset \mathbb{R}^X$, then it is unique.

2. Conversely, if two RKHS have the same r.k., then they are equal.

**Proof.** To prove 1., let $\mathcal{H}$ be a RKHS with two r.k. kernels $K$ and $K'$. For any two points $x, y \in X$, we need to show that $K(x, y) = K'(x, y)$. By the first property of RKHS, we know that the functions $K_x$ and $K'_x$ are in $\mathcal{H}$, and using the second property we obtain:

$$\|K_x - K'_x\|_{\mathcal{H}}^2 = \langle K_x - K'_x, K_x - K'_x \rangle_{\mathcal{H}}$$

$$= \langle K_x - K'_x, K_x \rangle_{\mathcal{H}} - \langle K_x - K'_x, K'_x \rangle_{\mathcal{H}}$$

$$= K_x(x) - K'_x(x) - K_x(x) + K'_x(x)$$

$$= 0.$$
$\mathcal{H}$ being a Hilbert space, only the zero function has a norm equal to 0. This shows that $K_x = K'_x$ as functions, and in particular that $K_x(y) = K'_x(y)$, i.e., $K(x, y) = K'(x, y)$.

To prove the converse, let us first consider a RKHS $\mathcal{H}_1$ with r.k. $K$. By definition of the r.k., we know that all the functions $K_x$ for $x \in X$ are in $\mathcal{H}_1$, therefore their linear span

$$\mathcal{H}_0 = \left\{ \sum_{i=1}^{n} \alpha_i K_{x_i} : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, x_1, \ldots, x_n \in X \right\}$$

is a subspace of $\mathcal{H}_1$. Now we observe that if $f \in \mathcal{H}_1$ is orthogonal to $\mathcal{H}_0$, then in particular it is orthogonal to $K_x$ for any $x$ which implies $f(x) = \langle f, K_x \rangle_{\mathcal{H}_1} = 0$, i.e., $f = 0$. In other words, $\mathcal{H}_0$ is dense in $\mathcal{H}_1$.

Moreover the $\mathcal{H}_1$ norm for functions in $\mathcal{H}_0$ only depends on the r.k. $K$, because it is given for a function $f = \sum_{i=1}^{n} \alpha_i K_{x_i} \in \mathcal{H}_0$ by

$$\| f \|_{\mathcal{H}_1}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \langle K_{x_i}, K_{x_j} \rangle_{\mathcal{H}_1} = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j K(x_i, x_j).$$

Suppose now that $\mathcal{H}_2$ is also a RKHS that admits $K$ as r.k. Then by the same argument, the space $\mathcal{H}_0$ is dense in $\mathcal{H}_2$, and the $\mathcal{H}_2$ norm in $\mathcal{H}_0$ is given by (3). In particular, for any $f \in \mathcal{H}_0$, $\| f \|_{\mathcal{H}_1} = \| f \|_{\mathcal{H}_2}$.

Let now $f \in \mathcal{H}_1$. By density of $\mathcal{H}_0$ in $\mathcal{H}_1$, there is a sequence $(f_n)$ in $\mathcal{H}_0$ such that $\| f_n - f \|_{\mathcal{H}_1} \to 0$. The converging sequence $(f_n)$ is in particular a Cauchy sequence for the $\mathcal{H}_1$ norm, and since this norm coincides with the $\mathcal{H}_2$ norm on $\mathcal{H}_0$, $(f_n)$ is also a Cauchy sequence for the $\mathcal{H}_2$ norm and converges in $\mathcal{H}_2$ to a function $g \in \mathcal{H}_2$. By Corollary 1 applied to both $\mathcal{H}_1$ and $\mathcal{H}_2$, we see that, for any $x \in X$, $\lim_{n \to +\infty} f_n(x) = f(x) = g(x)$. In other words, $f = g$ and therefore $f \in \mathcal{H}_2$. This shows that $\mathcal{H}_1 \subset \mathcal{H}_2$ and, by symmetry of the argument, in fact that $\mathcal{H}_1 = \mathcal{H}_2$. We now need to check that the norms in $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide, which results from:

$$\| f \|_{\mathcal{H}_1} = \lim_{n \to +\infty} \| f_n \|_{\mathcal{H}_1} = \lim_{n \to +\infty} \| f_n \|_{\mathcal{H}_2} = \| f \|_{\mathcal{H}_2}. \quad \square$$

References