ON ESTIMATION OF NONSMOOTH FUNCTIONALS OF SPARSE NORMAL MEANS

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Abstract. We study the problem of estimation of the value $N_\gamma(\theta) = \sum_{i=1}^d |\theta_i|^{\gamma}$ for $0 < \gamma \leq 1$ based on the observations $y_i = \theta_i + \varepsilon \xi_i$, $i = 1, \ldots, d$, where $\theta = (\theta_1, \ldots, \theta_d)$ are unknown parameters, $\varepsilon > 0$ is known, and $\xi_i$ are i.i.d. standard normal random variables. We prove that the non-asymptotic minimax risk on the class $B_0(s)$ of $s$-sparse vectors $\theta$ satisfies

$$\inf_{\hat{T}} \sup_{\theta \in B_0(s)} \mathbb{E}_\theta \left[ (\hat{T} - N_\gamma(\theta))^2 \right] \asymp \begin{cases} \varepsilon^2 s^2 \log^2(1 + d/s^2), & \text{if } s \leq \sqrt{d}, \\ \varepsilon^2 s^2 \log^{-\gamma}(1 + s^2/d), & \text{if } s > \sqrt{d}, \end{cases}$$

and we propose estimators achieving the minimax rate.

1. Introduction

In recent years, there has been a growing interest in statistical estimation of non-smooth functionals [1, 6, 13, 14, 7, 8, 2, 5]. Some of these papers deal with the normal means model [1, 2] addressing the problems of estimation of the $\ell_1$-norm and of the sparsity index, respectively. In the present paper, we analyze a family of non-smooth functionals including, in particular, the $\ell_1$-norm. We establish non-asymptotic minimax optimal rates of estimation on the classes of sparse vectors and we construct estimators achieving these rates.

Assume that we observe

$$y_i = \theta_i + \varepsilon \xi_i, \quad i = 1, \ldots, d,$$

where $\theta = (\theta_1, \ldots, \theta_d)$ is an unknown vector of parameters, $\varepsilon > 0$ is a known noise level, and $\xi_i$ are i.i.d. standard normal random variables. We consider the problem of estimating the functionals

$$N_\gamma(\theta) = \sum_{i=1}^d |\theta_i|^{\gamma}, \quad 0 < \gamma \leq 1,$$

assuming that the vector $\theta$ is $s$-sparse, that is, $\theta$ belongs to the class

$$B_0(s) = \{ \theta \in \mathbb{R}^d : \|\theta\|_0 \leq s \}.$$

Here, $\|\theta\|_0$ denotes the number of nonzero components of $\theta$ and $s \in \{1, \ldots, d\}$. We measure the accuracy of an estimator $\hat{T}$ of $N_\gamma(\theta)$ by the maximal quadratic risk over $B_0(s)$:

$$\sup_{\theta \in B_0(s)} \mathbb{E}_\theta \left[ (\hat{T} - N_\gamma(\theta))^2 \right].$$

Here and in the sequel, we denote by $\mathbb{E}_\theta$ the expectation with respect to the joint distribution $P_\theta$ of $(y_1, \ldots, y_d)$ satisfying (1).
In this paper, for all $0 < \gamma \leq 1$ we propose rate optimal estimators in a non-asymptotic minimax sense, that is, estimators $\hat{T}_\gamma^*$ such that
\[
\sup_{\theta \in B_0(s)} \mathbb{E}_\theta[(\hat{T}_\gamma^* - N_\gamma(\theta))^2] \asymp \inf_{T} \sup_{\theta \in B_0(s)} \mathbb{E}_\theta[(T - N_\gamma(\theta))^2],
\]
where $\inf_{T}$ denotes the infimum over all estimators and, for two quantities $a$ and $b$ possibly depending on $s,d,\varepsilon,\gamma$, we write $a \asymp b$ if there exist positive constants $c',c''$ that may depend only on $\gamma$ such that $c' \leq a/b \leq c''$. We also establish the following explicit non-asymptotic characterization of the minimax risk :
\[
R_{s,d}(\varepsilon,\gamma) := \inf_{T} \sup_{\theta \in B_0(s)} \mathbb{E}_\theta[(\hat{T} - N_\gamma(\theta))^2] \asymp \begin{cases} 
\varepsilon^2 s^2 \log(1 + d/s), & \text{if } s \leq \sqrt{d}, \\
\varepsilon^2 s^2 \log^\gamma(1 + s^2/d), & \text{if } s > \sqrt{d}.
\end{cases}
\]
Note that the rate on the right hand side of (2) is an increasing function of $s$, which is slightly greater than $\varepsilon^2 s^2$ for $s$ much smaller than $\sqrt{d}$, equal to $\varepsilon^2 s^2$ for $s \asymp \sqrt{d}$, and slightly smaller than $\varepsilon^2 s^2$ for $s$ much greater than $\sqrt{d}$.

In the case $s = d$, $\gamma = \varepsilon = 1$, the same minimax risk was studied in Cai and Low [1], where it was proved that
\[
R_{d,d}(1,1) = \inf_{T} \sup_{\theta \in \mathbb{R}^d} \mathbb{E}_\theta[(\hat{T} - N_1(\theta))^2] \asymp \frac{d^2}{\log d}
\]
and also claimed that $R_{s,d}(1,1) \asymp s^2/(\log d)$ for $s \geq d^\beta$ with $\beta > 1/2$, which agrees with (2).

We see from (2) that, for the general sparsity classes $B_0(s)$ and any $\gamma \in (0,1]$, there exist two different regimes with an elbow at $s \asymp \sqrt{d}$. We call them the sparse zone and the dense zone. The estimation methods for these two regimes are quite different. In the sparse zone, where $s$ is smaller than $\sqrt{d}$, we use that one can use suitably adjusted thresholding to achieve optimality. In this zone, rate optimal estimators can be obtained based on the techniques developed in [3] to construct minimax optimal estimators of linear and quadratic functionals. In the dense zone, where $s$ is greater than $\sqrt{d}$, we use another approach. We follow the general scheme of estimation of non-smooth functionals from [9] and our construction is essentially close in the spirit to [1]. Specifically, we consider the best polynomial approximation of the function $|x|^\gamma$ in a neighborhood of the origin and plug in unbiased estimators of the coefficients of this polynomial. Outside of this neighborhood, for $i$ such that $|y_i|$ is, roughly speaking, greater than the "noise level" of the order $\sqrt{\log d}$, we use $|y_i|^\gamma$ as an estimator of $|\theta_i|^\gamma$. The main difference from the estimator suggested in [1] for $\gamma = 1$ lies in the fact that, for the polynomial approximation part, we need to introduce a block structure with exponentially increasing blocks and carefully chosen thresholds depending on $s$. This is needed to achieve optimal bounds for all $s$ in the dense zone and not only for $s = d$ (or $s$ comfortably greater than $\sqrt{d}$).

This paper is organized as follows. In Section 2, we introduce the estimators and state the upper bounds for their risks. Section 3 provides the matching lower bounds. The rest of the paper is devoted to the proofs. In particular, some useful results from approximation theory are collected in Section 6.

2. Definition of estimators and upper bounds for their risks

In this section, we propose two different estimators, for the dense and sparse regimes defined by the inequalities $s^2 \geq 4d$ and $s^2 < 4d$, respectively. Recall that, in the Introduction, we used the inequalities $s \geq \sqrt{d}$ and $s < \sqrt{d}$, respectively, to define the two regimes. The factor 4
that we introduce in the definition here is a matter of convenience for the proofs. We note that such a change does not influence the final result since the optimal rate (cf. (2)) is the same, up to a constant, for all \( s \) such that \( s \ll \sqrt{d} \).

2.1. Dense zone: \( s^2 \geq 4d \). For any positive integer \( K \), we denote by \( P_{\gamma,K}(\cdot) \) the best approximation of \( |x|^\gamma \) by polynomials of degree at most \( 2K \) on the interval \([-1,1] \), that is

\[
\max_{x \in [-1,1]} \left| |x|^\gamma - P_{\gamma,K}(x) \right| = \min_{G \in \mathcal{P}_{2K}} \max_{x \in [-1,1]} \left| |x|^\gamma - G(x) \right|
\]

where \( \mathcal{P}_{K} \) is the class of all real polynomials of degree at most \( K \). Since \( |x|^\gamma \) is an even function, it suffices to consider approximation by polynomials of even degree. The quality of the best polynomial approximation of \( |x|^\gamma \) is described by Lemma 7 below.

We denote by \( a_{\gamma,2k} \) the coefficients of the canonical representation of \( P_{\gamma,K} \):

\[
P_{\gamma,K}(x) = \sum_{k=0}^{K} a_{\gamma,2k} x^{2k}, \quad x \in \mathbb{R},
\]

and by \( H_k(\cdot) \) the \( k \)th Hermite polynomial

\[
H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad k \in \mathbb{N}, \quad x \in \mathbb{R}.
\]

To construct the estimator in the dense zone, we use the sample duplication device, \( i.e. \), we transform \( y_i \) into randomized observations \( y_{1,i}, y_{2,i} \) as follows. Let \( z_1, \ldots, z_d \) be i.i.d. random variables such that \( z_i \sim \mathcal{N}(0, \sigma^2) \) and \( z_1, \ldots, z_d \) are independent of \( y_1, \ldots, y_d \). Set

\[
y_{1,i} = y_i + z_i, \quad y_{2,i} = y_i - z_i, \quad i = 1, \ldots, d.
\]

Then, \( y_{1,i} \sim \mathcal{N}(\theta_i, \sigma^2), y_{2,i} \sim \mathcal{N}(\theta_i, \sigma^2) \) for \( i = 1, \ldots, d \), where \( \sigma^2 = 2\varepsilon^2 \) and the random variables \( (y_{1,1}, \ldots, y_{1,d}; y_{2,1}, \ldots, y_{2,d}) \) are mutually independent.

Define the estimator of \( \gamma \) as follows:

\[
\hat{N}_\gamma = \sum_{i=1}^{d} \xi_\gamma(y_{1,i}, y_{2,i})
\]

where

\[
\xi_\gamma(u, v) = \sum_{l=0}^{L} \hat{P}_{\gamma,K_l} f_{\gamma}(u) I_{t_{l-1} < |v| \leq t_l} + |u|^\gamma I_{|v| > t_L},
\]

and

\[
\begin{align*}
\hat{P}_{\gamma,K_l} f_{\gamma}(u) &= \sum_{k=1}^{K_l} \sigma^{2k} a_{\gamma,2k} M_l^{\gamma-2k} H_{2k}(u/\sigma), \\
K_l &= 4^l c \log(s^2/d), \\
M_l &= 2^{l+1} \sqrt{2 \log(s^2/d)}, \\
t_l &= 2^{l} \sqrt{2 \log(s^2/d)}, \quad t_{-1} = 0,
\end{align*}
\]

(4)

Here and in what follows \( I_{\{\cdot\}} \) denotes the indicator function, and \( c > 0 \) is a constant that will be chosen small enough (see the proof of Theorem 1 below).
We will show that the estimator $\hat{N}_\gamma$ is optimal in a non-asymptotic minimax sense on the class $B_0(s)$ in the dense zone. The next theorem provides an upper bound on the risk of $\hat{N}_\gamma$ in this zone.

**Theorem 1.** Let the integers $d$ and $s \in \{1, \ldots, d\}$ be such that $s^2 \geq 4d$ and let $0 < \gamma \leq 1$. Then the estimator defined in (3) satisfies
\[
\sup_{\theta \in B_0(s)} \mathbb{E}_{\theta}[\left(\hat{N}_\gamma - N_\gamma(\theta)\right)^2] \leq C \frac{\varepsilon^{2\gamma} s^2}{\log^\gamma(s^2/d)},
\]
where $C > 0$ is a constant depending only on $\gamma$.

**2.2. Sparse zone:** $s^2 \leq 4d$. If $s$ belongs to the sparse zone we do not invoke the sample duplication and we use the estimator
\[
\hat{N}_\gamma^* = \sum_{i=1}^{d} \{ |y_i|^\gamma - \varepsilon^\gamma \alpha_\gamma \} \mathbb{1}_{y_i^2 > 2\varepsilon^2 \log(1 + d/s^2)},
\]
where \(\alpha_\gamma = \frac{E(|\xi|^{\gamma} \mathbb{1}_{\xi^2 > 2\varepsilon^2 \log(1 + d/s^2)})}{P(\xi^2 > 2\varepsilon^2 \log(1 + d/s^2))}\) for $\xi \sim N(0, 1)$.

The next theorem establishes an upper bound on the risk of this estimator.

**Theorem 2.** Let the integers $d$ and $s \in \{1, \ldots, d\}$ be such that $s^2 \leq 4d$ and $0 < \gamma \leq 1$. Then the estimator defined in (5) satisfies
\[
\sup_{\theta \in B_0(s)} \mathbb{E}_{\theta}[\left(\hat{N}_\gamma^* - N_\gamma(\theta)\right)^2] \leq C \varepsilon^{2\gamma} s^2 \log^\gamma(1 + d/s^2),
\]
where $C > 0$ is a constant depending only on $\gamma$.

Note that, intuitively, the optimal estimator in the sparse zone can be viewed as an example of applying the following routine developed in [3]. We start from the optimal estimator in the case $s = d$ and we threshold every term. Then, we center every term by its mean under the assumption that there is no signal. Finally, we choose a threshold that makes the best compromise between the first and second type errors in the support estimation problem. The only subtle ingredient in applying this argument in the present context is that we drop the polynomial part, which would almost always be removed by thresholding. In fact, one can notice that the polynomial approximation is only useful in a neighborhood of 0 but in the sparse zone we renounce to estimating small instances of $\theta_i$.

**3. Lower bounds**

We denote by $\mathcal{L}$ the set of all monotone non-decreasing functions $\ell : [0, \infty) \rightarrow [0, \infty)$ such that $\ell(0) = 0$ and $\ell \neq 0$.

**Theorem 3.** Let $s, d$ be integers such that $s \in \{1, \ldots, d\}$ and let $\ell(\cdot)$ be any loss function in the class $\mathcal{L}$. There exist positive constants $c_1$ and $c_2$ depending only on $\gamma$ and $\ell(\cdot)$ such that
\[
\inf_T \sup_{\theta \in B_0(s)} \mathbb{E}_{\theta} \ell\left(c_1 (\varepsilon^\gamma s \log^\gamma(1 + d/s^2))^{-1} |\hat{T} - N_\gamma(\theta)|\right) \geq c_2,
\]
where $\inf_T$ denotes the infimum over all estimators.
The proof follows the lines of the proof of the lower bound in [3, Theorem 1] with the only difference that \( L(\theta) = \sum_{i=1}^{d} \theta_i \) should be replaced by \( \sum_{i=1}^{d} \gamma_i \). Note that though Theorem 3 is valid for all \( s \in \{1, \ldots, d\} \) the bound becomes suboptimal in the dense zone.

**Theorem 4.** Let \( s, d \) be integers such that \( s \in \{1, \ldots, d\} \) and let \( \ell(\cdot) \) be any loss function in the class \( \mathcal{L} \). There exist positive constants \( c_1 \) and \( c_2 \) depending only on \( \gamma \) and \( \ell(\cdot) \) and a constant \( C \geq 4 \) depending only on \( \gamma \) such that, if \( s^2 \geq C d \), then

\[
\inf_{T} \sup_{\theta \in \mathcal{B}_0(s)} \mathbb{E}_{\theta} \ell \left( c_1 \left( \frac{e^{\gamma s}}{\log^2 (s^2/d)} \right)^{-1} |\hat{T} - N_{\gamma}(\theta)| \right) \geq c_2.
\]

where \( \inf_{T} \) denotes the infimum over all estimators.

In the case of quadratic loss \( \ell(u) = u^2 \), combining these two theorems with the bounds of Theorems 1 and 2, immediately leads to the relation (2).

4. **Proofs of the upper bounds**

Throughout the proofs, we denote by \( C \) positive constants that can depend only on \( \gamma \) and may take different values on different appearances.

4.1. **Proof of Theorem 1.** Denote by \( S \) the support of \( \theta \). We start with a bias-variance decomposition

\[
(\hat{N}_{\gamma} - N_{\gamma}(\theta))^2 \leq 4 \left( \sum_{i \in S} \mathbb{E}_{\theta} \xi_{\gamma}(y_{1,i}, y_{2,i}) - \sum_{i \in S} |\theta_i| \right)^2
\]

\[
+ 4 \left( \sum_{i \in S} \xi_{\gamma}(y_{1,i}, y_{2,i}) - \sum_{i \in S} \mathbb{E}_{\theta} \xi_{\gamma}(y_{1,i}, y_{2,i}) \right)^2
\]

\[
+ 4 \left( \sum_{i \notin S} \mathbb{E}_{\theta} \xi_{\gamma}(y_{1,i}, y_{2,i}) \right)^2
\]

\[
+ 4 \left( \sum_{i \notin S} \xi_{\gamma}(y_{1,i}, y_{2,i}) - \sum_{i \notin S} \mathbb{E}_{\theta} \xi_{\gamma}(y_{1,i}, y_{2,i}) \right)^2
\]

leading to the bound

\[
\mathbb{E}_{\theta}[(\hat{N}_{\gamma} - N_{\gamma}(\theta))^2] \leq 4s^2 \max_{i \in S} B_i^2 + 4s \max_{i \in S} V_i
\]

\[
+ 4d^2 \max_{i \notin S} B_i^2 + 4d \max_{i \notin S} V_i,
\]

where \( B_i = \mathbb{E}_{\theta} \xi_{\gamma}(y_{1,i}, y_{2,i}) - |\theta_i| \) is the bias of \( \xi_{\gamma}(y_{1,i}, y_{2,i}) \) as an estimator of \( |\theta_i| \) and \( V_i = \text{Var}_{\theta}(\xi_{\gamma}(y_{1,i}, y_{2,i})) \) is its variance. We now bound separately the four terms in (6).

1°. **Bias for \( i \notin S \).** If \( i \notin S \), then using Lemma 2 we obtain

\[
|B_i| = \sigma^2 \mathbb{E}[|\xi_{\gamma}| \mathbb{I}(|\xi| > t_L)] \leq C \sigma^2 e^{-t_L^2/2}, \quad \xi \sim \mathcal{N}(0, 1).
\]

The last exponential is smaller than \( 1/d \) by the definition of \( t_L \), so that

\[
d^2 \max_{i \notin S} B_i^2 \leq C \sigma^2 \gamma d \leq C \frac{\sigma^2 \gamma^2 s^2}{\log (s^2/d)}.
\]
Lemma 3 to obtain 

\[ (8) \quad V_i \leq \sum_{l=0}^{L} \mathbb{E} \hat{P}_{\gamma,K_l,M_l}(\sigma \xi) \mathbb{P}(|\xi| > t_{l-1}) + \sigma^{2\gamma} \mathbb{E} |\xi|^{2\gamma} \mathbb{P}(|\xi| > t_L), \quad \xi \sim \mathcal{N}(0,1). \]

The last term in (8) is bounded from above as in item 1°. Next, in view of Lemma 3,

\[ \mathbb{E} \hat{P}_{\gamma,K_0,M_0}(\sigma \xi) \leq C \sigma^{2\gamma} \frac{6^{2K_0}}{(M_0/\sigma)^2} \leq \frac{C \sigma^{2\gamma}}{\log(s^2/d)} \leq \frac{C \sigma^{2\gamma}s^2}{d \log(s^2/d)} \]

if \( c \) is chosen such that \( 2c \log 6 \leq 1 \). Here, we use the assumption \( s^2 \geq 4d \). For \( l \geq 1 \), we use Lemma 3 to obtain

\[ \mathbb{E} \hat{P}_{\gamma,K_l,M_l}(\sigma \xi) \mathbb{P}(|\xi| > t_{l-1}) \leq C \sigma^{2\gamma} \frac{6^{2K_l} e^{-t_{l-1}/2}}{(M_l/\sigma)^2} \leq \frac{C \sigma^{2\gamma} s^2}{4 \log(s^2/d)} \]

if we chose \( c \) such that \( 2c \log 6 \leq 1/4 \). In conclusion, under this choice of \( c \), using the facts that \( s^2 \geq 4d \) and \( 0 < \gamma \leq 1 \) we get

\[ (9) \quad d \max_{i \notin S} V_i \leq \frac{C \sigma^{2\gamma} s^2}{\log^2(s^2/d)}. \]

3°. Bias for \( i \in S \). If \( i \in S \), the bias has the form

\[ B_i = \sum_{l=0}^{L} \mathbb{E} \hat{P}_{\gamma,K_l,M_l}(X) \mathbb{P}(\sigma t_{l-1} < |X| \leq \sigma t_l) + \mathbb{E} |X|^\gamma \mathbb{P}(|X| > \sigma t_L) - |\theta_i|^\gamma, \]

where \( X \sim \mathcal{N}(\theta_i, \sigma^2) \). We will analyze this expression separately in three different ranges of values of \( |\theta_i| \).

3.1°. Case \( 0 < |\theta_i| < 2\sigma t_0 \). In this case, we use the bound

\[ |B_i| \leq \max_l |\mathbb{E} \hat{P}_{\gamma,K_l,M_l}(X) - |\theta_i|^\gamma| + \mathbb{E} |X|^\gamma - |\theta_i|^\gamma| \mathbb{P}(|X| > \sigma t_L), \]

where \( X \sim \mathcal{N}(\theta_i, \sigma^2) \). Since \( |\theta_i| \leq M_l \) for all \( l \), we can use Lemma 4 to obtain

\[ (10) \quad |\mathbb{E} \hat{P}_{\gamma,K_l,M_l}(X) - |\theta_i|^\gamma| \leq C \frac{M_l}{K_l} \gamma \leq \frac{C \sigma^\gamma}{\log^2(s^2/d)}. \]

In addition, using Lemma 1 we get

\[ |\mathbb{E} |X|^\gamma - |\theta_i|^\gamma| \mathbb{P}(|X| > \sigma t_L) \leq C \sigma^\gamma \mathbb{P}(|\xi| > t_L - |\theta_i|/\sigma) \leq \frac{C \sigma^\gamma}{\log(s^2/d)} \]

where \( \xi \sim \mathcal{N}(0,1) \) and we have used the inequalities \( t_L > 3t_0 \) and \( |\theta_i|/\sigma < 2t_0 \). It follows that

\[ (11) \quad s^2 \max_{0 < |\theta_i| < 2\sigma t_0} B_i^2 \leq \frac{C \sigma^{2\gamma}s^2}{\log^2(s^2/d)}. \]
3.2°. Case $2\sigma t_0 < |\theta| \leq 2\sigma t_L$. Let $l_0 \in \{1, \ldots, L - 1\}$ be the integer such that $\sigma t_0 < |\theta| \leq \sigma t_{l_0 + 1}$. We have

$$|B_i| \leq \sum_{l=0}^{l_0-1} \left| E\hat{P}_{\gamma,K_l,M_l}(X) - |\theta|^\gamma \cdot P(\sigma t_{l-1} < |X| \leq \sigma t_l) + \max_{l \geq l_0} |E\hat{P}_{\gamma,K_l,M_l}(X) - |\theta|^\gamma| + |E|X|^\gamma - |\theta|^\gamma|, $$

where $X \sim \mathcal{N}(\theta_i, \sigma^2)$. Analogously to (10) we find

$$\max_{l \geq l_0} |E\hat{P}_{\gamma,K_l,M_l}(X) - |\theta|^\gamma| \leq \frac{C|\sigma|^\gamma}{\log^{\gamma/2}(s^2/d)}. $$

Next, Lemma 1 and the fact that $|\theta| > 2\sigma t_0 = 2\sigma(2\log(s^2/d)/d)^{1/2}$ imply

$$|E|X|^\gamma - |\theta|^\gamma| \leq C|\sigma|(|\theta|/|\sigma t|)^{2-\gamma} \leq C|\sigma|^\gamma \log^{\gamma/2-1}(s^2/d) \leq \frac{C|\sigma|^\gamma}{\log^{\gamma/2}(s^2/d)}. $$

Finally, we consider the first sum on the right hand side of (12). Notice that

$$P(\sigma t_{l-1} < |X| \leq \sigma t_l) \leq e^{-\frac{s^2}{8c^2}}, \quad l = 0, \ldots, l_0 - 1, $$

since $|\theta| > \sigma t_0 \geq 2\sigma t_l$ for $l < l_0$. Using these inequalities and Lemma 5 we get

$$\sum_{l=0}^{l_0-1} \left| E\hat{P}_{\gamma,K_l,M_l}(X) \right| \cdot P(\sigma t_{l-1} < |X| \leq \sigma t_l) \leq C|\sigma|^\gamma \sum_{l=0}^{l_0-1} 6^{K_l} K_l^{3/2} e^{(c-1)|\theta|^2/(8\sigma^2)} \leq C|\sigma|^\gamma \sum_{l=0}^{l_0-1} t_l^3 e^{(c \log 6 + c-1)t_l^2/2}. $$

Choose $c > 0$ such that $c \log 6 + c < 1/4$. As $t_l = 2^l \sqrt{2 \log(s^2/d)}$, this yields

$$\sum_{l=0}^{l_0-1} \left| E\hat{P}_{\gamma,K_l,M_l}(X) \right| \cdot P(\sigma t_{l-1} < |X| \leq \sigma t_l) \leq C|\sigma|^\gamma e^{-(1/2) \log(s^2/d)/d} \leq \frac{C|\sigma|^\gamma}{\log^{\gamma/2}(s^2/d)}. $$

Furthermore,

$$\sum_{l=0}^{l_0-1} |\theta|^\gamma P(\sigma t_{l-1} < |X| \leq \sigma t_l) \leq l_0 |\theta|^\gamma e^{-\frac{s^2}{8c^2}} \leq C \log \left( \frac{\theta^2}{2\sigma^2 \log(s^2/d)} \right) |\theta|^\gamma e^{-\frac{s^2}{8c^2}} \leq C|\sigma|^\gamma e^{-\frac{s^2}{16c^2}}.
where we have used that $|\theta_i| > \sigma t_0 = \sigma 2^{b_0} \sqrt{\log(s^2/d)}$. Since $l_0 \geq 1$, this also implies that (14) does not exceed

$$\frac{C\sigma^\gamma}{\log^{1/2}(s^2/d)} \leq \frac{C\sigma^\gamma}{\log^{1/2}(s^2/d)}.$$  

Combining the above arguments yields

$$s^2 \max_{2\sigma t_0 < |\theta_i| \leq 2\sigma t_L} B_i^2 \leq \frac{C\sigma^\gamma s^2}{\log(s^2/d)}.$$  

3.3°. Case $|\theta_i| > 2\sigma t_L$. Recall that the bias $B_i$ has the form

$$B_i = \sum_{l=0}^{L} E\tilde{P}_{\gamma, K_l, M_l}(X) P(\sigma t_{l-1} < |X| \leq \sigma t_l) + E|X|^\gamma P(|X| > \sigma t_L) - |\theta_i|^{\gamma},$$

where $X \sim N(\theta_i, \sigma^2)$. Using Lemma 5 we get

$$\left| \sum_{l=0}^{L} E\tilde{P}_{\gamma, K_l, M_l}(X) P(\sigma t_{l-1} < |X| \leq \sigma t_l) \right| \leq \max_{l=0,\ldots,L} \left| E\tilde{P}_{\gamma, K_l, M_l}(X) P(|X| \leq \sigma t_L) \right| \leq C\sigma^\gamma 6^{K_L} K_L^{3/2} e^{t_0^2/(8\sigma^2)} e^{-t_0^2/(8\sigma^2)} \leq C\sigma^\gamma (\log d)^{3/2} 6^{9c} \log d e^{9(c-1) \log d}$$

and the last upper bound is smaller than $C\sigma^\gamma \log^{-\gamma/2}(s^2/d)$ if $c > 0$ is small enough. On the other hand, it follows from (13) that $|E|X|^\gamma - |\theta_i|^{\gamma|} \leq C\sigma^\gamma \log^{-\gamma/2}(s^2/d)$. Thus,

$$\left| E|X|^\gamma P(|X| > \sigma t_L) - |\theta_i|^{\gamma} \right| \leq \left| E|X|^\gamma - |\theta_i|^{\gamma|} \right| + |\theta_i|^{\gamma} P(|X| \leq \sigma t_L) \leq C\sigma^\gamma \log^{-\gamma/2}(s^2/d) + |\theta_i|^{\gamma} e^{-\frac{\gamma^2}{8\sigma^2}} \leq C\sigma^\gamma \log^{-\gamma/2}(s^2/d).$$

Finally, we get

$$s^2 \max_{|\theta_i| > 2\sigma t_L} B_i^2 \leq \frac{C\sigma^\gamma s^2}{\log(s^2/d)}.$$  

4°. Variance for $i \in S$. We consider the same three cases as in item 3° above. For the first two cases, it suffices to use a coarse bound granting that, for all $i \in S$,

$$V_i \leq E\theta[\ell_0^2(y_{1,i}, y_{2,i})] = \sum_{l=0}^{L} E\tilde{P}_{\gamma, K_l, M_l}(X) P(\sigma t_{l-1} < |X| \leq \sigma t_l) + E|X|^{2\gamma} P(|X| > \sigma t_L)$$

where $X \sim N(\theta_i, \sigma^2)$.

4.1°. Case $0 < |\theta_i| < 2\sigma t_0$. In this case, we deduce from (17) that

$$V_i \leq \max_{l=0,\ldots,L} E\tilde{P}_{\gamma, K_l, M_l}(X) + E|X|^{2\gamma},$$
where $X \sim N(\theta, \sigma^2)$. Lemma 4 and the fact that $E|X|^{2\gamma} \leq \sigma^{2\gamma} + |\theta|^{|2\gamma}$ imply

$$V_i \leq CM_L^{2\gamma} 2^{K \log L} + \sigma^{2\gamma} + |\theta|^{|2\gamma}$$

$$\leq C\sigma^{2\gamma} \log^2(d) \sigma^{2c\log^2 + C\sigma^{2\gamma} \log(s^2/d)}.$$

Hence, if $c > 0$ is small enough, we conclude that

$$s \max_{0 < |\theta| < 2\sigma t_0} V_i \leq \frac{C\sigma^{2\gamma} s^2}{\log^2(s^2/d)}.$$  

4.2. Case $2\sigma t_0 < |\theta| < 2\sigma t_L$. As in item 3.2 above, we denote by $l_0 \in \{1, \ldots, L - 1\}$ the integer such that $\sigma t_0 < |\theta| \leq \sigma t_{l_0 + 1}$. We deduce from (17) that

$$V_i \leq \max_{l=0, \ldots, l_0 - 1} \mathbf{E} \hat{P}_{\gamma, K, M_l}(X) P(|X| \leq \sigma t_{l_0 - 1}) + \max_{l=l_0, \ldots, L} \mathbf{E} \hat{P}_{\gamma, K, M_l}(X) + E|X|^{2\gamma},$$

where $X \sim N(\theta, \sigma^2)$. The last two terms on the right hand side are controlled as in item 4.1. For the first term, we find using Lemma 5 that, for $X \sim N(\theta, \sigma^2)$,

$$\max_{l=0, \ldots, l_0 - 1} \mathbf{E} \hat{P}_{\gamma, K, M_l}(X) P(|X| \leq \sigma t_{l_0 - 1})$$

$$\leq C\sigma^{2\gamma} (\sigma/M_0)^{1-2\gamma} 6^{2K_{l_0 - 1}} e^{(\log(1 + 4/c)\theta^2) / (4\sigma^2)} e^{-\theta^2 / (8\sigma^2)}$$

$$\leq C\sigma^{2\gamma} \log^{-1}(s^2/d) e^{(c\log 6 + 4c\log(1 + 4/c) - 1/2)^{t_{l_0 - 1}}}.$$

Choosing $c > 0$ small enough allows us to obtain the desired bound

$$s \max_{2\sigma t_0 < |\theta| < 2\sigma t_L} V_i \leq \frac{C\sigma^{2\gamma} s^2}{\log^2(s^2/d)}.$$  

4.3. Case $|\theta| > 2\sigma t_L$. We first note that

$$\mathbf{Var}(|y_{1,i}|^\gamma 1_{|y_{2,i}| > \sigma t_L}) = P(|X| > \sigma t_L) \mathbf{Var}(|X|^\gamma) + (E|X|^\gamma)^2 P(|X| \leq \sigma t_L)$$

$$\leq C[\sigma^{2\gamma} + |\theta|^{|2\gamma} P(|X| \leq \sigma t_L)],$$

where $X \sim N(\theta, \sigma^2)$ and $\mathbf{Var}(|X|^\gamma) \leq C\sigma^{2\gamma}$ by Lemma 1 while $(E|X|^\gamma)^2 \leq E|X|^{2\gamma} \leq \sigma^{2\gamma} + |\theta|^{|2\gamma}$. Using this remark we obtain

$$V_i \leq 2\mathbf{Var} \left( \sum_{i=0}^{L} \hat{P}_{\gamma, K, M_l}(y_{1,i}) 1_{\sigma t_{l-1} < |y_{2,i}| \leq \sigma t_l} \right) + 2\mathbf{Var} \left( |y_{1,i}|^\gamma 1_{|y_{2,i}| > \sigma t_L} \right)$$

$$\leq 2 \sum_{i=0}^{L} \mathbf{E} \hat{P}_{\gamma, K, M_l}(X) P(|X| \leq \sigma t_{l-1} < |X| \leq \sigma t_l) + C[\sigma^{2\gamma} + |\theta|^{|2\gamma} P(|X| \leq \sigma t_L)]$$

$$\leq C \left( \max_{l=0, \ldots, L} \mathbf{E} \hat{P}_{\gamma, K, M_l}(X) P(|X| \leq \sigma t_L) + \sigma^{2\gamma} + |\theta|^{|2\gamma} P(|X| \leq \sigma t_L) \right).$$

Here, the term

$$\max_{l=0, \ldots, L} \mathbf{E} \hat{P}_{\gamma, K, M_l}(X) P(|X| \leq \sigma t_L)$$

is controlled via an argument analogous to (19) while

$$|\theta|^{|2\gamma} P(|X| \leq \sigma t_L) \leq |\theta|^{|2\gamma} e^{-\frac{\theta^2}{8\sigma^2}} \leq C\sigma^{2\gamma}. $$
This allows us to conclude that

\[ s \max_{|\theta_i| > 2s \sigma L} V_i \leq \frac{C \sigma^2 \gamma^2 s^2}{\log^2(s^2/d)}. \]

The result of the theorem follows now from (6), (7), (9), (11), (15), (16), (18), (20), and (21).

4.2. Proof of Theorem 2. Denoting by \( S \) the support of \( \theta \) we have

\[
\hat{N}_\gamma^* - N_\gamma(\theta) = \sum_{i \in S} \{ |y_i|^{\gamma} - \varepsilon \gamma \alpha \gamma - |\theta_i|^{\gamma} \} - \sum_{i \in S} \{ |y_i|^{\gamma} - \varepsilon \gamma \alpha \gamma \} \mathbb{1}_{y_i^2 \leq 2\varepsilon^2 \log(1+d/s^2)}
\]

\[+ \sum_{i \notin S} \{ |y_i|^{\gamma} - \varepsilon \gamma \alpha \gamma \} \mathbb{1}_{y_i^2 > 2\varepsilon^2 \log(1+d/s^2)}, \]

so that

\[
E_{\theta} \left[ (\hat{N}_\gamma^* - N_\gamma(\theta))^2 \right] \leq 4E_{\theta} \left( \sum_{i \in S} \{ |y_i|^{\gamma} - |\theta_i|^{\gamma} \} \right)^2 + 2^{\gamma^2 \varepsilon^2} s^2 \log^2(1 + d/s^2)
\]

\[+ 4 \varepsilon^2 s^2 \alpha \gamma^2 + 4d \varepsilon^2 \gamma \mathbb{E} \left[ (|\xi|^{\gamma} - \alpha \gamma) \mathbb{1}_{\xi^2 > 2\log(1+d/s^2)} \right]\]

where \( \xi \sim \mathcal{N}(0, 1) \). Using Lemma 1 we get

\[
E_{\theta} \left( \sum_{i \in S} \{ |y_i|^{\gamma} - |\theta_i|^{\gamma} \} \right)^2 \leq C \varepsilon^2 s^2.
\]

Next, by the Hölder inequality and the standard bounds on the tails of the Gaussian distribution,

\[ \alpha \gamma \leq \left( \frac{E(|\xi| \mathbb{1}_{\xi^2 > 2\log(1+d/s^2)})}{\mathbb{P}(\xi^2 > 2\log(1+d/s^2))} \right)^{\gamma} \leq C \log^{\gamma/2}(1 + d/s^2) \]

for all \( s \) satisfying \( s^2 \leq 4d \). For such \( s \), we also have

\[
E \left[ (|\xi|^{\gamma} - \alpha \gamma)^2 \mathbb{1}_{\xi^2 > 2\log(1+d/s^2)} \right] \leq 2E(\xi^{2\gamma} \mathbb{1}_{\xi^2 > 2\log(1+d/s^2)}) + 2\alpha^2 \gamma \mathbb{P}(\xi^2 > 2\log(1+d/s^2))
\]

\[\leq C \log^{-1}(1 + d/s^2) E(\xi^{2\gamma} \mathbb{1}_{\xi^2 > 2\log(1+d/s^2)})
\]

\[+ C \log^2(1 + d/s^2) \mathbb{P}(\xi^2 > 2\log(1+d/s^2)) \]

\[\leq C(s^2/d \log(1 + d/s^2)), \]

again due to the standard bounds on the tails of the Gaussian distribution. Combining the above inequalities proves the theorem.

5. Lemmas for the proof of Theorem 1

Lemma 1. If \( X \sim \mathcal{N}(\theta, \sigma^2) \) with \( \theta \neq 0 \), and \( \gamma \in (0, 1) \), then

\[ |E|X|^{\gamma} - |\theta|^{\gamma}| \leq C \sigma^2 \min \left\{ 1, \left( \frac{\sigma}{|\theta|} \right)^{2-\gamma} \right\}, \]

\[ \text{Var}(|X|^{\gamma}) \leq C \sigma^{2\gamma}. \]
Lemma 2. Let $\vartheta \in \mathbb{R}$ and $X \sim \mathcal{N}(\vartheta, 1)$. For any $k \in \mathbb{N}$, the $k$-th Hermite polynomial satisfies

$$
\mathbb{E}H_k(X) = \vartheta^k,
$$

$$
\mathbb{E}H_k^2(X) \leq k^k(1 + \vartheta^2/k)^k.
$$

The proof of this lemma can be found in [1].
Lemma 3. Let $\hat{P}_{\gamma,K,M}$ be defined in (4) with parameters $K = K_l$ and $M = M_l$ for some $l \in \{0, \ldots, L\}$ and small enough $c > 0$. If $X \sim \mathcal{N}(0, \sigma^2)$, then

$$E\hat{P}^2_{\gamma,K,M}(X) \leq C\sigma^{2\gamma} \frac{6^{2K}}{(M/\sigma)^2},$$

where $C > 0$ is a constant depending only on $\gamma$.

Proof. Recall that, for the Hermite polynomials, $E(H_k(\xi)H_j(\xi)) = 0$ if $k \neq j$ and $\xi \sim \mathcal{N}(0, 1)$. Using this fact and then Lemmas 8 and 2 we obtain

$$E\hat{P}^2_{\gamma,K,M}(X) = M^{2\gamma} \sum_{k=1}^{K} a_{\gamma,2k}(\sigma/M)^{4k} E H^2_{2k}(X/\sigma) \leq C 6^{2K} M^{2\gamma} \sum_{k=1}^{K} (2k)^{2k}(\sigma/M)^{4k}.$$

Moreover, since $\sigma^2/M^2 = c/(8K)$ we have

$$\sum_{k=1}^{K} (2k)^{2k}(\sigma/M)^{4k} \leq 4\frac{\sigma^4}{M^4} + \sum_{2k \leq \log(M/\sigma)} (\sigma/M)^{4k}(2\log(M/\sigma))^{2k} + \sum_{\log(M/\sigma) < k \leq K} (c/4)^{2k} \leq C \frac{\sigma^4}{M^4}$$

if $c$ is small enough. We conclude that

$$E\hat{P}^2_{\gamma,K,M}(X) \leq C\sigma^{2\gamma} \frac{6^{2K}}{(M/\sigma)^{4-2\gamma}} \leq C\sigma^{2\gamma} \frac{6^{2K}}{(M/\sigma)^2}.$$

Lemma 4. Let $\hat{P}_{\gamma,K,M}$ be defined in (4) with parameters $K = K_l$ and $M = M_l$ for some $l \in \{0, \ldots, L\}$ and small enough $c > 0$. If $X \sim \mathcal{N}(\vartheta, \sigma^2)$ with $|\vartheta| \leq M$, then

$$|E\hat{P}_{\gamma,K,M}(X) - |\vartheta|^\gamma| \leq C \left(\frac{M}{K}\right)^\gamma,$$

$$E\hat{P}^2_{\gamma,K,M}(X) \leq CM^{2\gamma}2^{8K},$$

where $C > 0$ is a constant depending only on $\gamma$.

Proof. To prove the first inequality of the lemma, it is enough to note that, due to Lemma 2,

$$E\hat{P}_{\gamma,K,M}(X) = \sum_{k=1}^{K} a_{\gamma,2k} M^{\gamma-2k} \vartheta^{2k}$$

and to apply Lemma 7. For the second inequality, we use the bound

$$E\hat{P}^2_{\gamma,K,M}(X) \leq M^{2\gamma} \left( \sum_{k=1}^{K} \sigma^{2k} |a_{\gamma,2k}| M^{-2k} \sqrt{E H^2_{2k}(X/\sigma)} \right)^2.$$
Thus Lemmas 8 and 2 together with the relations $|\vartheta| \leq M$ and $K = (c/8)M^2/\sigma^2$ imply that, for small enough $c > 0$,

$$
\mathbb{E} \hat{P}_{\gamma,K,M}^2(X) \leq CM^{2\gamma}6^{2K} \left( \sum_{k=1}^{K} M^{-2k}(2M^2)^{k} \right)^2 \leq CM^{2\gamma}2^{8K}.
$$

\[ \square \]

**Lemma 5.** Let $\hat{P}_{\gamma,K,M}$ be defined in (4) with parameters $K = K_l$ and $M = M_l$ for some $l \in \{0, \ldots, L\}$ and small enough $c > 0$. If $X \sim N(\vartheta, \sigma^2)$ with $|\vartheta| > 2\sigma t_l$, then

$$
\left| \mathbb{E} \hat{P}_{\gamma,K,M}(X) \right| \leq C\sigma\gamma6^{K}K^{3/2}e^{c/8\sigma^2},
$$

$$
\mathbb{E} \hat{P}_{\gamma,K,M}^2(X) \leq C\sigma\gamma(\sigma/M)^{4-2\gamma}6^{2K}e^{c\log(1+4/c)/4\sigma^2},
$$

where $C > 0$ is a constant depending only on $\gamma$.

**Proof.** To prove the first inequality of the lemma, we use (24) and Lemma 8 to obtain

$$
\left| \mathbb{E} \hat{P}_{\gamma,K,M}(X) \right| \leq C\sigma\gamma6^{K}K^{3/2}e^{c/8\sigma^2}.
$$

Recall that $M^2 = 8\sigma^2K/c$ and $|\vartheta| > M$ by assumption of the lemma. Thus,

$$
\sigma^{\gamma}6^{K}K^{3/2}e^{c/8\sigma^2} \leq C\sigma^{\gamma}6^{K}K^{3/2}e^{K\log(\vartheta^2/M^2)}
$$

and the result follows since $K\log(\vartheta^2/M^2) = cM^2/8\sigma^2\log(\vartheta^2/M^2) \leq c\vartheta^2/8\sigma^2$.

We now prove the second inequality of the lemma. Using (25) and then Lemmas 8 and 2 we get

$$
\mathbb{E} \hat{P}_{\gamma,K,M}^2(X) \leq CM^{2\gamma}6^{2K} \left( \sum_{k=1}^{K} (\sigma/M)^{2k}(2k)^{k} \right)^2.
$$

As $M^2 = 8\sigma^2K/c$ and $|\vartheta| > M$, we have

$$
\frac{\vartheta^2}{2\sigma^2k} \geq \frac{M^2}{2\sigma^2K} = \frac{4}{c} \geq 2
$$

for $c > 0$ small enough. Using this remark and the fact that the function $x \rightarrow x^{-1}\log(1+x)$ is decreasing for $x \geq 2$ we obtain

$$
k\log \left( 1 + \frac{\vartheta^2}{2\sigma^2k} \right) \leq \frac{c\log(1+4/c)\vartheta^2}{8\sigma^2}.
$$

Therefore,

$$
\mathbb{E} \hat{P}_{\gamma,K,M}^2(X) \leq CM^{2\gamma}6^{2K}e^{c\log(1+4/c)\vartheta^2/4\sigma^2} \left( \sum_{k=1}^{K} (\sigma/M)^{2k}(2k)^{k} \right)^2.
$$

Finally, the result follows by noticing that, by an argument analogous to (23), we have

$$
\sum_{k=1}^{K} (\sigma/M)^{2k}(2k)^{k} \leq \frac{C\sigma^2}{M^2}.
$$

\[ \square \]
6. SOME FACTS FROM APPROXIMATION THEORY

We start with a proposition relating moment matching to best polynomial approximation. It is similar to several results used in the theory of estimation of non-smooth functionals starting from Lepski et al. [9]. There exist different techniques of proving such results for specific examples. Thus, the proof in [9] is based on Riesz representation of linear operators, while Wu and Yang [14] provide an explicit construction using Lagrange interpolation. Here, for completeness we give a short proof for a relatively general setting based on optimization arguments.

Let \( f : [-1, 1] \rightarrow \mathbb{R} \) be a continuous even function. Consider the accuracy of best polynomial approximation of \( f \):

\[
\delta_K(f) = \inf_{\mu \in \mathcal{P}_K} \max_{x \in [-1, 1]} |f(x) - G(x)|
\]

where \( \mathcal{P}_K \) is the class of all real polynomials of degree at most \( K \).

**Proposition 1.** Let \( f : [-1, 1] \rightarrow \mathbb{R} \) be a continuous even function. For any even integer \( K \geq 1 \), there exist two probability measures \( \tilde{\mu}_0 \) and \( \tilde{\mu}_1 \) on \([-1, 1]\) such that

(i) \( \tilde{\mu}_0 \) and \( \tilde{\mu}_1 \) are symmetric about 0;
(ii) \( \int \mu_0(dt) = \int \mu_1(dt) \) for \( l = 0, 1, \ldots, K \);
(iii) \( \int f(t) \mu_0(dt) - \int f(t) \mu_0(dt) = 2\delta_K(f) \).

**Proof.** Denote by \( P_{\text{sym}} \) the set of all probability measures on \([-1, 1]\) that are symmetric about 0, and by \( P_2 \) be the set of all signed measures on \([-1, 1]\) with total variation not greater than 2. For \( K = 2m \), we have

\[
(26) \sup_{(\nu_0, \nu_1) \in P_{\text{sym}} \times P_{\text{sym}}} |\int f(x) d\nu_0(x) - \int f(x) d\nu_1(x)| = \sup_{\mu \in P_2} \left| \frac{1}{2} \int_{-1}^{1} f(x) d\mu(x) \right|
\]

where the third equality follows from Sion’s minimax theorem, and the second equality uses the fact that \( f \) is an even function, so that the maximum over \( \mu \in P_2 \) in the second line of (26) is equal to the maximum over symmetric \( \mu \in P_2 \) satisfying the same moment constraints. Let \((\nu_0^*, \nu_1^*)\) be the pair of probability measures attaining the maximum in the first line of (26). The proposition follows by setting \( \tilde{\mu}_i = \nu_i^*, \ i = 0, 1 \).

As an immediate corollary of Proposition 1 for \( f(x) = |x|^\gamma \), we obtain the following result.

**Lemma 6.** For any even integer \( K \geq 1 \) and any \( M > 0 \), there exist two probability measures \( \tilde{\mu}_0 \) and \( \tilde{\mu}_1 \) on \([-M, M]\) such that
(i) \( \tilde{\mu}_0 \) and \( \tilde{\mu}_1 \) are symmetric about 0;
(ii) \( \int t^l \tilde{\mu}_0(dt) = \int t^l \tilde{\mu}_1(dt) \) for \( l = 0, 1, \ldots, K \);
(iii) \( \int |t|^\gamma \tilde{\mu}_0(dt) - \int |t|^\gamma \tilde{\mu}_1(dt) = 2M^\gamma \delta_{K,\gamma} \).

For the function \( f(x) = |x|^\gamma \), the asymptotically exact behavior of the best polynomial approximation \( \delta_{K,\gamma} \) as \( K \to \infty \) is well known, see, for example, [11, Theorem 7.2.2] implying the following lemma.

**Lemma 7.** There exist positive constants \( c_* \) and \( C^* \) depending only on \( \gamma \) such that
\[
c_*K^{-\gamma} \leq \delta_{K,\gamma} \leq C^*K^{-\gamma}, \quad \forall \ K \in \mathbb{N}.
\]

Finally, the next lemma provides a useful bound on the coefficients \( a_{\gamma,2k} \) in the canonical representation of the polynomial of best approximation
\[
P_{\gamma,K}(x) = \sum_{k=0}^{K} a_{\gamma,2k}x^{2k}, \quad x \in \mathbb{R}.
\]

**Lemma 8.** Let \( P_{\gamma,K}(\cdot) \) be the polynomial of best approximation of degree \( 2K \) for \( |x|^\gamma \) on \([-1,1]\). Then the coefficients \( a_{\gamma,2k} \) in (27) satisfy
\[
|a_{\gamma,2k}| \leq C6^K, \quad k = 0, \ldots, K,
\]
where \( C > 0 \) is a constant depending only on \( \gamma \).

This lemma is an immediate corollary of the following more general fact, which is a consequence of Szegő’s theorem on the minimal eigenvalue of a lacunary version of the Hilbert matrix.

**Proposition 2.** Let \( P(x) = \sum_{k=0}^{N} a_kx^k \) be a polynomial such that \( |P(x)| \leq 1 \) for all \( x \in [-1,1] \). Then there exists an absolute constant \( C > 0 \) such that
\[
|a_k| \leq C(\sqrt{2} + 1)^N
\]
for all \( k \in \{0, \ldots, N\} \).

**Proof.** We have
\[
\int_{-1}^{1} \left( \sum_{k=0}^{N} a_kx^k \right)^2 dx = 2 \sum_{i,j=0}^{N} a_ia_j \mathbb{1}_{i+j \text{ even}}.
\]
It is easy to see that the quadratic form in (28) is positive definite for all \( N \). Furthermore, as shown by Szegő [10], the minimal eigenvalue \( \lambda_{\min}(N) \) of this quadratic form satisfies
\[
\lambda_{\min}(N) = 2^{9/4} \pi^{3/2} N^{1/2}(\sqrt{2} - 1)^{2N+3}(1 + o(1)) \quad \text{as } N \to \infty.
\]
Therefore, there exists an absolute constant \( C_0 > 0 \) such that \( \lambda_{\min}(N) \geq C_0(\sqrt{2} - 1)^{2N} \) for all \( N \). This inequality and (28) imply that
\[
C_0(\sqrt{2} - 1)^{2N} \sum_{k=0}^{N} a_k^2 \leq 1
\]
and hence \( \max_{k=0, \ldots, N} |a_k| \leq C_0^{1/2}(\sqrt{2} - 1)^{-N} \). \( \square \)
7. Construction of the priors for the proof of Theorem 4

The proof of Theorem 4 will be based on Theorem 2.15 in [12]. It proceeds by bounding the minimax risk from below by the Bayes risk with the prior measures on θ that we are going to define in this section.

In what follows we set

\[ \Lambda = \sqrt{\log \left( \frac{s^2}{d} \right)} , \quad M = \sigma \Lambda , \]

and we denote by K the smallest even integer such that

\[ K \geq \frac{3}{2} e \log \left( \frac{s^2}{d} \right) = \frac{3}{2} e \Lambda^2 . \]

We will also write for brevity

\[ B = B_0(s) . \]

In what follows, unless stated otherwise, \( \tilde{\mu}_0 \) and \( \tilde{\mu}_1 \) are the probability measures satisfying Lemma 6 where \( M \) is defined in (29) and \( K \) is the smallest even integer for which (30) holds.

For \( i = 0, 1 \), the probability measure \( \mu_i \) is defined as the distribution of random vector \( \theta \in \mathbb{R}^d \) with components \( \theta_j \) having the form \( \theta_j = \epsilon_j \eta_j , j = 1, \ldots, d \), where \( \epsilon_j \) is a Bernoulli random variable with \( P(\epsilon_j = 1) = \frac{s}{2d} \), \( \eta_j \) is distributed according to \( \tilde{\mu}_i \), and \( (\epsilon_1, \ldots, \epsilon_d, \eta_1, \ldots, \eta_d) \) are mutually independent.

Let \( \mathbb{P}_0 \) and \( \mathbb{P}_1 \) be the mixture probability measures defined by

\[ \mathbb{P}_i(A) = \int_{\mathbb{R}^d} \mathbb{P}_\theta(A) \mu_i( d\theta ), \quad i = 0, 1 , \]

for any measurable set \( A \). The densities of \( \mathbb{P}_0 \) and \( \mathbb{P}_1 \) with respect to the Lebesgue measure on \( \mathbb{R}^d \) are given by

\[ f_0(x) = \prod_{i=1}^d h(x_i) \quad \text{and} \quad f_1(x) = \prod_{i=1}^d g(x_i) , \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d , \]

respectively, where for \( x \in \mathbb{R} \) we set

\[ h(x) = \frac{s}{2d} \phi_0(x) + \left( 1 - \frac{s}{2d} \right) \phi(x) \]

and

\[ g(x) = \frac{s}{2d} \phi_1(x) + \left( 1 - \frac{s}{2d} \right) \phi(x) \]

with

\[ \phi_i(x) = \int_{\mathbb{R}} \phi(x-t) \tilde{\mu}_i( dt ), \quad i = 0, 1 , \]

where we denote by \( \phi(\cdot) \) the density of the \( \mathcal{N}(0, \sigma^2) \) distribution.

Note that the measures \( \mu_0 \) and \( \mu_1 \) are not supported in \( B \). We associate to them two probability measures \( \mu_{0,B} \) and \( \mu_{1,B} \) supported in \( B \) and the corresponding mixture measures defined by

\[ \mu_{i,B}(A) = \frac{\mu_i(A \cap B)}{\mu_i(B)}, \quad \mathbb{P}_{i,B}(A) = \int_{\mathbb{R}^d} \mathbb{P}_\theta(A) \mu_{i,B}( d\theta ), \quad i = 0, 1 , \]

for any measurable set \( A \).
8. Proof of Theorem 4

Since we have \( \ell(t) \geq \ell(a)1_{t>a} \) for any \( a > 0 \), it is enough to prove the theorem for the indicator loss \( \ell(t) = 1_{t>a} \).

Furthermore, since rescaling by a constant does not change the result, we will assume that the model is \( y_i = \theta_i + \sigma \xi_i \) rather than \( y_i = \theta_i + \varepsilon_i \) (recall that \( \sigma = \sqrt{2\varepsilon} \)).

Introduce the following notation:

\[
m_i = \int_{\real^d} N_\gamma(\theta)\mu_i(d\theta), \quad v_i^2 = \int_{\real^d} (N_\gamma(\theta) - m_i)^2 \mu_i(d\theta), \quad i = 0, 1.
\]

Note that Lemmas 6 and 7 imply:

\[
(32) \quad m_1 - m_0 = d\left( \int_{\real^d} |\theta_1|^\gamma \mu_1(d\theta) - \int_{\real^d} |\theta_1|^\gamma \mu_0(d\theta) \right) = \frac{s}{2} \left( \int_{-1}^1 |t|^\gamma \tilde{\mu}_1(dt) - \int_{-1}^1 |t|^\gamma \tilde{\mu}_0(dt) \right)
\]

\[
= sM^\gamma \delta_{K,\gamma} \geq c_s s(M/K)^\gamma \geq C_1 \frac{\sigma^\gamma s}{\Lambda^\gamma},
\]

where \( C_1 > 0 \) is a constant depending only on \( \gamma \).

Let \( V(P, Q) \) denote the total variation distance between two probability measures \( P \) and \( Q \). For any \( u > 0 \) and any \( c \in \real \) we have, using Theorem 2.15 in [12],

\[
(33) \quad \inf_{\theta} \sup_{\theta \in B_0(s)} P_\theta(|T - N_\gamma(\theta)| \geq u) \geq \frac{1 - V'}{2},
\]

where

\[
V' = V(P_{0,B}, P_{1,B}) + \mu_0(B(N_\gamma(\theta) \geq c) + \mu_1,B(N_\gamma(\theta) \leq c + 2u).
\]

We now apply (33) with the parameters

\[
c = m_0 + 3v_0, \quad u = \frac{m_1 - m_0}{4}.
\]

By Chebyshev-Cantelli inequality,

\[
(34) \quad \mu_0(N_\gamma(\theta) \geq c) \leq \frac{v_0^2}{v_0^2 + (c - m_0)^2} = \frac{1}{10}.
\]

Next, we easily get

\[
\max(v_0^2, v_1^2) \leq dM^2\gamma = d\sigma^2\gamma \Lambda^2\gamma.
\]

Thus, we may write

\[
\max(v_0, v_1) \leq \left( \frac{\sqrt{\sigma}}{s} \Lambda^2\gamma \right) \frac{\sigma^\gamma s}{\Lambda^\gamma},
\]

where, for \( \tilde{C} \) large enough, \( \frac{\sqrt{\sigma}}{s} \Lambda^2\gamma = \frac{\sqrt{\sigma}}{s} \log^{\gamma}(\frac{x^2}{\tilde{C}}) \leq C_1/12 \). Therefore,

\[
(35) \quad \max(v_0, v_1) \leq \frac{C_1 \sigma^\gamma s}{12 \Lambda^\gamma}.
\]

It follows from (32), (35) and Chebyshev-Cantelli inequality that

\[
(36) \quad \mu_1(N_\gamma(\theta) \leq c + 2u) = \mu_1(N_\gamma(\theta) - m_1 \leq - \frac{m_1 + m_0}{2} + 3v_0) \leq \mu_1(N_\gamma(\theta) - m_1 \leq - \frac{m_1 - m_0}{2} + 3v_0) \leq \mu_1(N_\gamma(\theta) - m_1 \leq - \frac{C_1 \sigma^\gamma s}{4 \Lambda^\gamma}) \leq \frac{1}{10}.
\]
By Lemma 9, we have $\mu_i(B) \geq 7/8$, $i = 0, 1$. Combining these inequalities with (34) and (36) we immediately conclude that

$$(37) \quad \mu_0(B) N_\gamma(\theta) \geq c + \mu_1(B) N_\gamma(\theta) \leq 8/35.$$ 

Next, we consider the total variation distance $V(\mathbb{P}_{0,B}, \mathbb{P}_{1,B})$. Using Lemma 9 we get that, for $\bar{C}$ large enough,

$$(38) \quad V(\mathbb{P}_{0,B}, \mathbb{P}_{1,B}) \leq V(\mathbb{P}_{0,B}, \mathbb{P}_0) + V(\mathbb{P}_0, \mathbb{P}_1) + V(\mathbb{P}_1, \mathbb{P}_{1,B})$$

$$\leq V(\mathbb{P}_0, \mathbb{P}_1) + \mu_0(B^c) + \mu_1(B^c)$$

$$\leq \sqrt{\chi^2(\mathbb{P}_1, \mathbb{P}_0)/2} + 1/4$$

$$\leq (\sqrt{2} + 1)/4,$$

where the last two inequalities are due to Pinsker’s inequality and Lemma 11, respectively. Combining (33), (37) and (38) we get that, if $s^2 \geq \bar{C}d$ for $\bar{C} > 0$ large enough, there exists a constant $C > 0$ depending only on $\gamma$ such that

$$\inf_{\hat{T}} \sup_{\theta \in B_0(s)} \mathbb{P}_\theta(|\hat{T} - N_\gamma(\theta)| \geq C \sigma s / \Lambda) > \frac{1}{16}.$$ 

This completes the proof.

9. LEMMAS FOR THE PROOF OF THEOREM 4

**Lemma 9.** For $i = 0, 1$, we have

$$V(\mathbb{P}_i, \mathbb{P}_{i,B}) \leq \mu_i(B^c).$$

Furthermore, there exists an absolute constant $\bar{C} > 0$ such that, for any $s^2 \geq \bar{C}d$,

$$\mu_i(B^c) \leq 1/8, \quad i = 0, 1.$$ 

The proof of this lemma is quite standard. For example, repeating the argument of Lemma 4 in [4] we get that $V(\mathbb{P}_i, \mathbb{P}_{i,B}) \leq \mu_i(B^c) = \mathbb{P}(B(d, \frac{\theta}{d}) > s) \leq e^{-\frac{s^2}{2d}}$. Here, $B(d, \frac{\theta}{d})$ is the binomial random variable with parameters $d$ and $\frac{\theta}{d}$.

**Lemma 10.** Let $\hat{\mu}_0$ and $\hat{\mu}_1$ be two probability measures on $[-M, M]$ satisfying the moment matching property (ii) of Lemma 6 with some $K \geq 1$. Let $\phi_0$ and $\phi_1$ be defined in (31) where $\phi$ is the density of $\mathcal{N}(0, \sigma^2)$ distribution. Then

$$\int \frac{(\phi_0(x) - \phi_1(x))^2}{\phi(x)} dx \leq \sum_{k=K+1}^{\infty} \Lambda^{2k} k!$$

where $\Lambda = M/\sigma$. 

Proof. By rescaling, it suffices to consider the case \( \sigma = 1 \), \( M = \Lambda \). Introducing the notation \( E_i(k) = \int t^k \tilde{\mu}_i(dt) \), \( i = 0, 1 \), it is straightforward to check that
\[
\int \frac{(\phi_0(x) - \phi_1(x))^2}{\phi(x)}\,dx = \int e^{g\varphi} \tilde{\mu}_1(d\varphi) + \int e^{g\varphi} \tilde{\mu}_0(d\varphi') - 2 \int e^{g\varphi} \tilde{\mu}_1(d\varphi) \tilde{\mu}_0(d\varphi')
\]
\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \bigg( (E_1(k))^2 + (E_0(k))^2 - 2E_1(k)E_0(k) \bigg)
\]
\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \bigg( E_1(k) - E_0(k) \bigg)^2.
\]

It remains to notice that \( E_1(k) = E_0(k) \) for \( k = 0, \ldots, K \), by property (ii) of Lemma 6, and \( |E_1(k) - E_0(k)| \leq 2^k \) for all \( k \). \( \square \)

Lemma 11. If \( s^2 \geq 4d \), then
\[
\chi^2(\mathbb{P}_1, \mathbb{P}_0) < 1/4.
\]

Proof. Since \( \mathbb{P}_0 \) and \( \mathbb{P}_1 \) are product measures we have
\[
\chi^2(\mathbb{P}_1, \mathbb{P}_0) = \left( 1 + \int \frac{(g-h)^2}{h} \, dx \right) - 1,
\]
cf., e.g., [12, page 86]. It follows from the definition of \( g \) and \( h \) and from Lemma 10 that
\[
\int \frac{(g-h)^2}{h} \leq \frac{1}{2d} \left( \frac{s}{2d} \right)^2 \int \frac{(\phi_1 - \phi_0)^2}{\phi} \leq 2 \left( \frac{s}{2d} \right)^2 \sum_{k=K+1}^{\infty} \frac{2^{2k}}{k!}.
\]

Using the inequalities \( k! \geq (k/e)^k \) and \( 1 + x \leq e^x \) we get
\[
\chi^2(\mathbb{P}_1, \mathbb{P}_0) \leq \exp \left( \frac{s^2}{2d} \sum_{k=K+1}^{\infty} \left( \frac{e\Lambda^2}{k} \right)^k \right) - 1.
\]

Recall that \( K \geq 3\varepsilon\Lambda^2/2 \) and \( 2 \leq 3\varepsilon\Lambda^2/2 \). Thus,
\[
\frac{s^2}{2d} \sum_{k=K+1}^{\infty} \left( \frac{e\Lambda^2}{k} \right)^k \leq \frac{s^2}{2d} \sum_{k=K+1}^{\infty} (2/3)^k = \frac{s^2}{d} (2/3)^K < \frac{4s^2}{d^2} \exp \left( 3\varepsilon \log(2/3) L^2/2 \right) = \frac{4}{9} \left( \frac{s^2}{d} \right)^a
\]
where \( a = 1 + 3\varepsilon \log(2/3)/2 < -0.6 \). Since \( s^2 \geq 4d \) we get \( \chi^2(\mathbb{P}_1, \mathbb{P}_0) \leq \exp(4^{0.4}/9) - 1 < 1/4. \) \( \square \)

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