MVA "Kernel methods"
Homework 1

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Exercice 1. Combining kernels.

1. Let K_1 and K_2 be two positive definite (p.d.) kernels on a set $\mathcal{X}$. Show that the functions $K_1 + K_2$ and $K_1 \times K_2$ are also p.d. on $\mathcal{X}$.
2. Let $(K_i)_{i\geq 1}$ a sequence of p.d. kernel on a set $\mathcal{X}$ such that, for any $(x,y) \in \mathcal{X}^2$, the sequence $(K_i(x,y))_{i\geq 0}$ be convergent. Show that the pointwise limit:

$$K(x,y) = \lim_{i\to+\infty} K_i(x,y)$$

is also p.d. (assuming the limit exists for any x, y).
3. Show that the following kernel is p.d.:

$$\forall x, y \in \mathbb{R} \quad K(x,y) = 3^{xy}\,.$$

Exercice 2. Completeness of the RKHS.

We want to finish the construction of the RKHS associated to a positive definite kernel K given in the course. Remember we have defined the set of functions:

$$\mathcal{H}_0 = \left\{ \sum_{i=1}^{n} \alpha_i K_{x_i} \,:\, n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, x_1, \ldots, x_n \in \mathcal{X} \right\}$$

and for any two functions $f, g \in \mathcal{H}_0$, given by:

$$f = \sum_{i=1}^{m} a_i K_{\mathbf{x}_i}, \quad g = \sum_{j=1}^{n} b_j K_{\mathbf{y}_j}\,,$$

we have defined the operation:

$$\langle f, g \rangle_{\mathcal{H}_0} := \sum_{i,j} a_i b_j K(\mathbf{x}_i, \mathbf{y}_j)\,.$$

In the course we have shown that $\mathcal{H}_0$ endowed with this inner product is a pre-Hilbert space. Let us now show how to finish the construction of the RKHS from $\mathcal{H}_0$

1. Show that any Cauchy sequence (f_n) in $\mathcal{H}_0$ converges pointwisely to a function $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = \lim_{n\to+\infty} f_n(x)$.

2. Show that any Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}_0$ which converges pointwise to 0 satisfies:

$$\lim_{n\to+\infty} \| f_n \|_{\mathcal{H}_0} = 0\,.$$

3. Let $\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$ be the set of functions $f : \mathcal{X} \to \mathbb{R}$ which are pointwise limits of Cauchy sequences in $\mathcal{H}_0$, i.e., if (f_n) is a Cauchy sequence in $\mathcal{H}_0$, then $f(x) = \lim_{n\to+\infty} f_n(x)$. Show that $\mathcal{H}_0 \subset \mathcal{H}$.

4. If (f_n) and (g_n) are two Cauchy sequences in $\mathcal{H}_0$, which converge pointwisely to two functions f and $g \in \mathcal{H}$, show that the inner product $\langle f_n, g_n \rangle_{\mathcal{H}_0}$ converges to a number which only depends on f and g. This allows us to define formally the operation:

$$\langle f, g \rangle_{\mathcal{H}} = \lim_{n\to+\infty} \langle f_n, g_n \rangle_{\mathcal{H}_0}\,.$$

5. Show that $\langle .,. \rangle_{\mathcal{H}}$ is an inner product on $\mathcal{H}$.

6. Show that $\mathcal{H}_0$ is dense in $\mathcal{H}$ (with respect to the metric defined by the inner product $\langle .,. \rangle_{\mathcal{H}}$)

7. Show that $\mathcal{H}$ is complete.

8. Show that $\mathcal{H}$ is a RKHS whose reproducing kernel is K.

Exercice 3. Uniqueness of the RKHS

Prove that if $K : \mathcal{X} \times \mathcal{X}$ is a positive definite function, then it is the r.k. of a unique RKHS. To prove it, you can consider two possible RKHS $\mathcal{H}$ and $\mathcal{H}'$, and show that (i) they contain the same elements and (ii) their inner products are the same. (Hint: consider the linear space spanned by the functions $K_x : t \mapsto K(x,t)$, and use the fact that a linear subspace $\mathcal{F}$ of a Hilbert space $\mathcal{H}$ is dense in $\mathcal{H}$ if and only 0 is the only vector orthgonal to all vectors in $\mathcal{F}$)