MVA "Kernel methods in machine learning" Homework 1

Julien Mairal and Jean-Philippe Vert

Upload your answers (in PDF) to:
`http://goo.gl/xcC4Qy`
before January 25, 2017, 1pm (Paris time).

Exercice 1. Kernel examples
Are the following kernels positive definite?
1.

$$\forall x, y \in \mathbb{R} \quad K_1(x, y) = 10^{xy}, \quad K_2(x, y) = 10^{x+y}.$$

2.

$$\forall x, y \in [0, 1) \quad K_3(x, y) = -\log(1 - xy).$$

3. Let $\mathcal{X}$ be a set and $f, g : \mathcal{X} \to \mathbb{R}_+$ two non-negative functions:

$$\forall x, y \in \mathcal{X} \quad K_4(x, y) = \min(f(x)g(y), f(y)g(x))$$

Exercice 2. Combining kernels
1. For $x, y \in \mathbb{R}$, let

$$K_1(x, y) = (xy + 1)^2 \quad \text{and} \quad K_2(x, y) = (xy - 1)^2.$$

What is the RKHS of K_1? Of K_2? Of $K_1 + K_2$?
2. Let K_1 and K_2 be two positive definite kernels on a set $\mathcal{X}$, and α, β two positive scalars. Show that $\alpha K_1 + \beta K_2$ is positive definite, and describe its RKHS.

Exercice 3. Uniqueness of the RKHS

Prove that if $K : \mathcal{X} \times \mathcal{X}$ is a positive definite function, then it is the r.k. of a unique RKHS. To prove it, you can consider two possible RKHS $\mathcal{H}$ and $\mathcal{H}'$, and show that (i) they contain the same elements and (ii) their inner products are the same. (Hint: consider the linear space spanned by the functions $K_x : t \mapsto K(x,t)$, and use the fact that a linear subspace $\mathcal{F}$ of a Hilbert space $\mathcal{H}$ is dense in $\mathcal{H}$ if and only 0 is the only vector orthgonal to all vectors in $\mathcal{F}$)