"Kernel methods in machine learning" Homework 1 Due January 20, 2021, 3pm

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Exercice 1. Kernels

Study whether the following kernels are positive definite:

1. $\mathcal{X} = \mathbb{N}, \quad K(x, x') = 2^{x+x'}$
2. $\mathcal{X} = \mathbb{R}, \quad K(x, x') = \cos(x + x')$
3. $\mathcal{X} = \mathbb{R}, \quad K(x, x') = \cos(x - x')$

Exercice 2. Function and kernel boundedness

Consider a p.d. kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $K(x, z) \leq b^2$ for all x, z in $\mathcal{X}$. Show that $\|f\|_\infty = \sup_{x \in \mathcal{X}} |f(x)| \leq b$ for any function f in the unit ball of the corresponding RKHS.

Exercice 3. Non-expansiveness of the Gaussian kernel

Consider the Gaussian kernel $K : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ such that for all pair of points x, x' in $\mathbb{R}^p$,

$$K(x, x') = e^{-\frac{\alpha}{2}\|x - x'\|^2},$$

where $\|.\|$ is the Euclidean norm on $\mathbb{R}^p$. Call $\mathcal{H}$ the RKHS of K and consider its RKHS mapping $\varphi : \mathbb{R}^p \to \mathcal{H}$ such that $K(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{H}}$ for all x, x' in $\mathbb{R}^p$. Show that

$$\|\varphi(x) - \varphi(x')\|_{\mathcal{H}} \leq \sqrt{\alpha}\|x - x'\|.$$

The mapping is called non-expansive whenever $\alpha \leq 1$.