"Kernel methods in machine learning"
Homework 2
Due February 3, 2021, 3pm

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Exercice 1.

Let $\mathcal{X}$ be a set and $\mathcal{F}$ be a Hilbert space. Let $\Psi : \mathcal{X} \to \mathcal{F}$, and $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be:

$$\forall x, x' \in \mathcal{X}, \quad K(x, x') = \langle \Psi(x), \Psi(x') \rangle_{\mathcal{F}} \, .$$

Show that K is a positive definite kernel on $\mathcal{X}$, and describe its RKHS $\mathcal{H}$.
(Hint: Show that any function of the form $f_w(x) = \langle \Psi(x), w \rangle_{\mathcal{F}}$ is in $\mathcal{H}$, for $w \in \mathcal{F}$, and explicit its norm.)

Exercice 2.

Prove that for any p.d. kernel K on a space $\mathcal{X}$, a function $f : \mathcal{X} \to \mathbb{R}$ belongs to the RKHS $\mathcal{H}$ with kernel K if and only if there exists $\lambda > 0$ such that $K(\mathbf{x}, \mathbf{x}') - \lambda f(\mathbf{x}) f(\mathbf{x}')$ is p.d.
(Hint: you can use the result of Exercice 5.1. that we discussed in the course.)