Uniqueness of the RKHS

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Recall the definition of an RKHS:

Definition 1. *Let $\mathcal{X}$ be a set and $\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$ be a class of functions forming a Hilbert space with inner product $\langle .,. \rangle_{\mathcal{H}}$. The function $K : \mathcal{X}^2 \mapsto \mathbb{R}$ is called a* reproducing kernel (r.k.) *of $\mathcal{H}$ if*

1. *$\mathcal{H}$ contains all functions of the form*

$$\forall \mathbf{x} \in \mathcal{X}, \quad K_{\mathbf{x}} : \mathbf{t} \mapsto K(\mathbf{x}, \mathbf{t}) \,. \tag{1}$$

2. *For every $\mathbf{x} \in \mathcal{X}$ and $f \in \mathcal{H}$ the* reproducing property *holds:*

$$f(\mathbf{x}) = \langle f, K_{\mathbf{x}} \rangle_{\mathcal{H}} \,. \tag{2}$$

If a r.k. exists, then $\mathcal{H}$ is called a reproducing kernel Hilbert space (RKHS).

Remember that an RKHS has the following property

Theorem 1. *A Hilbert space of functions $\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$ is a RKHS if and only if for any $\mathbf{x} \in \mathcal{X}$, the mapping $f \mapsto f(\mathbf{x})$ (from $\mathcal{H}$ to $\mathbb{R}$) is continuous.*

Suppose a sequence of function $(f_n)_{n\in\mathbb{N}}$ converges in a RKHS to a function $f \in \mathcal{H}$. Then the functions $(f_n - f)$ converges to 0 in the RKHS sense, from which we deduce that $f_n(x) - f(x)$ also converges to 0 for any $x \in \mathcal{X}$, by continuity of the evaluations functionals. This proves that:

Corollary 1. *Convergence in a RKHS implies pointwise convergence on any point, i.e., if f_n converges to $f \in \mathcal{H}$, then $f_n(x)$ converges to $f(x)$ for any $x \in \mathcal{X}$.*

We now detail the proof of the following result, due to **?**, which shows that there is a one-to-one correspondance between RKHS and r.k. It allows us to talk about "the" RHKS associated to a r.k., and conversely to "the" r.k. associated to a RKHS.

Theorem 2. 1. *If a r.k. exists for a Hilbert space $\mathcal{H} \subset \mathbb{R}^{\mathcal{X}}$, then it is unique.*

2. *Conversely, if two RKHS have the same r.k., then they are equal.*

Proof. To prove 1., let $\mathcal{H}$ be a RKHS with two r.k. kernels K and K'. For any two points $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we need to show that $K(\mathbf{x}, \mathbf{y}) = K'(\mathbf{x}, \mathbf{y})$. By the first property of RKHS, we know that the functions $K_{\mathbf{x}}$ and $K'_{\mathbf{x}}$ are in $\mathcal{H}$, and using the second property we obtain:

$$\begin{aligned}
\| K_{\mathbf{x}} - K'_{\mathbf{x}} \|_{\mathcal{H}}^2 &= \left\langle K_{\mathbf{x}} - K'_{\mathbf{x}}, K_{\mathbf{x}} - K'_{\mathbf{x}} \right\rangle_{\mathcal{H}} \\
&= \left\langle K_{\mathbf{x}} - K'_{\mathbf{x}}, K_{\mathbf{x}} \right\rangle_{\mathcal{H}} - \left\langle K_{\mathbf{x}} - K'_{\mathbf{x}}, K'_{\mathbf{x}} \right\rangle_{\mathcal{H}} \\
&= K_{\mathbf{x}}(\mathbf{x}) - K'_{\mathbf{x}}(\mathbf{x}) - K_{\mathbf{x}}(\mathbf{x}) + K'_{\mathbf{x}}(\mathbf{x}) \\
&= 0 \,.
\end{aligned}$$

$\mathcal{H}$ being a Hilbert space, only the zero function has a norm equal to 0. This shows that $K_{\mathbf{x}} = K'_{\mathbf{x}}$ as functions, and in particular that $K_{\mathbf{x}}(\mathbf{y}) = K'_{\mathbf{x}}(\mathbf{y})$, i.e., $K(\mathbf{x},\mathbf{y}) = K'(\mathbf{x},\mathbf{y})$.
To prove the converse, let us first consider a RKHS $\mathcal{H}_1$ with r.k. K. By definition of the r.k., we know that all the functions K_x for $x \in \mathcal{X}$ are in $\mathcal{H}_1$, therefore their linear span

$$\mathcal{H}_0 = \left\{ \sum_{i=1}^{n} \alpha_i K_{x_i} \,:\, n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, x_1, \ldots, x_n \in \mathcal{X} \right\}$$

is a subspace of $\mathcal{H}_1$. Now we observe that if $f \in \mathcal{H}_1$ is orthogonal to $\mathcal{H}_0$, then in particular it is orthogonal to K_x for any x which implies $f(x) = \langle f, K_x \rangle_{\mathcal{H}_1} = 0$, i.e., $f = 0$. In other words, $\mathcal{H}_0$ is dense in $\mathcal{H}_1$. Moreover the $\mathcal{H}_1$ norm for functions in $\mathcal{H}_0$ only depends on the r.k. K, because it is given for a function $f = \sum_{i=1}^{n} \alpha_i K_{x_i} \in \mathcal{H}_0$ by

$$\begin{aligned} \| f \|_{\mathcal{H}_1}^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \left\langle K_{x_i}, K_{x_j} \right\rangle_{\mathcal{H}_1} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j K(x_i, x_j) \,. \end{aligned} \tag{3}$$

Suppose now that $\mathcal{H}_2$ is also a RKHS that admits K as r.k. Then by the same argument, the space $\mathcal{H}_0$ is dense in $\mathcal{H}_2$, and the $\mathcal{H}_2$ norm in $\mathcal{H}_0$ is given by (3). In particular, for any $f \in \mathcal{H}_0$, $\| f \|_{\mathcal{H}_1} = \| f \|_{\mathcal{H}_2}$. Let now $f \in \mathcal{H}_1$. By density of $\mathcal{H}_0$ in $\mathcal{H}_1$, there is a sequence (f_n) in $\mathcal{H}_0$ such that $\| f_n - f \|_{\mathcal{H}_1} \to 0$. The converging sequence (f_n) is in particular a Cauchy sequence for the $\mathcal{H}_1$ norm, and since this norm coincides with the $\mathcal{H}_2$ norm on $\mathcal{H}_0$, (f_n) is also a Cauchy sequence for the $\mathcal{H}_2$ norm and converges in $\mathcal{H}_2$ to a function $g \in \mathcal{H}_2$. By Corollary 1 applied to both $\mathcal{H}_1$ and $\mathcal{H}_2$, we see that, for any $x \in \mathcal{X}$, $\lim_{n \to +\infty} f_n(x) = f(x) = g(x)$. In other words, $f = g$ and therefore $f \in \mathcal{H}_2$. This shows that $\mathcal{H}_1 \subset \mathcal{H}_2$ and, by symmetry of the argument, in fact that $\mathcal{H}_1 = \mathcal{H}_2$. We now need to check that the norms in $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide, which results from:

$$\| f \|_{\mathcal{H}_1} = \lim_{n \to +\infty} \| f_n \|_{\mathcal{H}_1} = \lim_{n \to +\infty} \| f_n \|_{\mathcal{H}_2} = \| f \|_{\mathcal{H}_2} \,.$$

□